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On the characteristic functions for extreme value distributions

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Abstract For the first time, explicit closed forms are derived for characteristic functions for the extreme value distributions of type 2 and type 3. These expressions involve the Fox's $H_{0,2}^{2,0}$ function and the Wright generalized confluent hypergeometric ${}_1\Psi_0$-function. A discussion of applications is given.

Keywords Characteristic function · Extreme value distribution of type 2 · Extreme value distribution of type 3 · Fox's H function · Wright generalized hypergeometric Ψ function

AMS 2000 Subject Classifications Primary—60E10; Secondary—33C60, 62G32

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed sequence of random variables, and let $M_n = \max_{1 \le k \le n} X_k$ denote the partial maximum. If there exist normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and a nondegenerate distribution $F(x)$ such that

$$\mathsf{P}\left(M_n \le a_n x + b_n\right) \longrightarrow F(x)$$

as $n \to \infty$ then $F(x)$ is said to be an *extreme value distribution* (EVD) (Fisher and Tippett 1928; Gnedenko 1943; Gumbel 1958). Three forms are possible for

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$F(x)$, known as EVDs of type 1, type 2 and type 3. In general form, the cumulative distribution function (CDF) of the EVD of type 1 (also known as Gumbel distribution) is

$$F(x) = \exp\Big\{ - \exp\Big(- \frac{x-\mu}{\sigma}\Big)\Big\} \tag{1.1}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$. The type 2 EVD (also known as Fréchet distribution) has the CDF

$$F(x) = \exp\Big\{ - \Big(\frac{x-\mu}{\sigma}\Big)^{-\alpha}\Big\} \tag{1.2}$$

for $x > \mu$, $\alpha > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$. The type 3 EVD has the CDF

$$F(x) = \exp\Big\{ - \Big(\frac{\mu-x}{\sigma}\Big)^{\alpha}\Big\}$$

for $x < \mu$, $\alpha > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$. The type 3 EVD is equivalent to the Weibull distribution that has the CDF

$$F(x) = 1 - \exp\Big\{ - \Big(\frac{x-\mu}{\sigma}\Big)^{\alpha}\Big\} \tag{1.3}$$

for $x > \mu$, $\alpha > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$.

The parameter, α, in Eqs. 1.2 and 1.3 controls the shape of the EVDs. The type 2 EVD has heavy upper tails for all $\alpha > 0$. Its upper tails become heavier with smaller values of α. The type 3 EVD has heavy upper tails for $0 < \alpha < 1$. Its upper tails become heavier with smaller values of $\alpha \in (0, 1)$. The type 3 EVD has exponentially decaying light upper tails for all $\alpha \geq 1$. Its upper tails become lighter with larger values of $\alpha \in [1, \infty)$. Since the EVDs of type 2 and type 3 accommodate heavy upper tails, they have received applications in numerous areas. These include: insurance claim-size distribution, estimation of extreme rainfall return levels, planetary perturbations on Oort cloud comets, modeling high-resolution synthetic aperture radar images, fatal traffic accidents, texture analysis, value-at-risk analysis, flood risk assessment, commodity price distribution, internet traffic and risk theory. For other application areas, we refer the readers to the excellent book Embrechts et al. (1997).

The characteristic function (CHF) of a random variable, X say, defined by $\phi_X(t) = \mathsf{E}\exp\{\mathrm{i}tX\}$, where $\mathrm{i} = \sqrt{-1}$, is a fundamental tool in probability and statistics. So, one would like to have a closed form expression for $\phi_X(t)$ for every X if that is possible. If X has the CDF given by Eq. 1.1 then it is well known that

$$\phi_X(t) = \exp\big\{\mathrm{i}\mu t\big\}\Gamma\big(1 - \mathrm{i}\sigma t\big). \tag{1.4}$$

However, for the distributions given by Eqs. 1.2 and 1.3, closed form expressions for $\phi_X(t)$ have not been known in the literature.

The CHFs can be used, for example, to derive the distribution of $X_1+X_2+\cdots+X_n$ when X_i, $i = 1, 2, \ldots, n$ are independent and follow the EVDs. The result in Eq. 1.4 has been used, for example, to derive the distribution of the sum of independent Gumbel random variables, see Nadarajah (2007, 2008a).

Sums of independent Fréchet random variables arise in a variety of contexts:

1. as models for the distribution of ground state energy in the context of disordered systems of statistical mechanics, see Biroli et al. (2007a, b).
2. as compound sums in insurance for the distribution of total claim amount.

Each of the cited papers contains explicit details involving sums of independent Fréchet random variables.

Sums of independent Weibull random variables are of prime importance in wireless communications and related areas, see Filho and Yacoub (2006). They also arise in a wide variety of other contexts:

1. for cell migration and proliferation during monolayer formation and wound healing (Roberts and Wiihem 1964).
2. as models for the long-term stress range response distribution in offshore structural reliability analysis (Veritas 1995).
3. study of the influence of publication delays on the aging of scientific literature (Egghe and Rousseau 2000).
4. as models for radar clutter and thermal noise (Helstrom 2000).
5. for spectral estimation in time series analysis (Gunter 2001).
6. as models for sums of waiting times (Kleiner 2001).
7. for modeling value at risk-efficient portfolios (Malevergne and Sornette1 2004).
8. for modeling rough surfaces (Yu and Polycarpou 2004).
9. as models to fit the distribution of sea clutter with spikes (Dong 2006).
10. as models for synthetic aperture radar images (Sun and Han 2009; Wan-She and Zheng 2000).

Each of the cited papers contains explicit details involving sums of independent Weibull random variables.

The authors are not aware of any work deriving expressions or even approximations for the distribution of sums of independent Fréchet random variables. However, there is considerable work giving approximations for the distribution of sums of independent Weibull random variables. Beaulieu (1990) provides an infinite series approximation for the distribution of the sum of Rayleigh random variables (Rayleigh is the particular case of Weibull for $\alpha = 2$). Hu and Beaulieu (2005) provide simple and accurate closed-form approximations to Rayleigh sum distributions and densities. Karagiannidis et al. (2005) propose a closed form upper bound for the distribution of the weighted sum of Rayleigh random variables. Vegas-Sanchez-Ferrero et al. (2010) suggest that a gamma distribution is a good approximation for the distribution of the weighted sum of Rayleigh random variables. Various authors have proposed approximations based on mixtures of exponentials for the distribution of the sum of Weibull random variables, see, for example, Feldmann and Whitt (1998), Asmussen (2000), Dufresne (2007) and Ko and Ng (2007).

Zhang (1999) gives the simplest approach for finding the Rayleigh sum distribution. The idea is to use the CHF. The CHF of a Rayleigh random variable directly involves the standard normal CDF, a well known function. So, by the inversion theorem, the Rayleigh sum distribution is a single integral of a well known function. Such integrals can be computed easily using any platform. This approach circumvents the need for approximations.

The motivation for our work comes from Zhang (1999). We first find expressions for the CHFs of Fréchet and Weibull random variables, see Sections 2 and 3. We then apply the inversion theorem for these expressions to find the sum distributions, see Section 4.

For the EVD of type 2, the authors are not aware of any work giving closed form expressions for $\phi_X(t)$. However, closed form expressions are possible for some particular cases. For example, if $\alpha = 1$ in Eq. 1.2 then $\phi_X(t)$ can be expressed in terms of the modified Bessel function of the second kind (Gradshteyn and Ryzhik 2000, Sections 8.407 and 8.43).

For the EVD of type 3, there has been considerable work to find closed form expressions for $\phi_X(t)$ for some particular cases. If α is an integer in Eq. 1.3 then Cheng et al. (2004) express $\phi_X(t)$ in terms of the Meijer's G function (Gradshteyn and Ryzhik 2000, Section 9.3). If α is a rational number then Sagias and Karagiannidis (2005) express $\phi_X(t)$ in terms of the Meijer's G function. Nadarajah and Kotz (2007) provide an alternative expression involving finite series of generalized hypergeometric functions (Gradshteyn and Ryzhik 2000, Section 9.14) for the case α is a rational number. Ismail and Matalgah (2007) use Padé approximation techniques to obtain closed-form approximations for $\phi_X(t)$. Muraleedharan et al. (2007) and Muraleedharan (2008) give two forms $\phi_X(t)$: the first is simply a re-expression of the definition of $\phi_X(t)$:

$$\phi_X(t) = \int_{-\infty}^{\infty} \cos(tx) f_X(x) dx + \mathrm{i} \int_{-\infty}^{\infty} \sin(tx) f_X(x) dx,$$

where $f_X(x)$ denotes the probability density function (PDF) of X. The second form given is grossly incorrect as pointed by Nadarajah (2008b, c). Finally, note that $\phi_X(t)$ takes an elementary form if $\alpha = 1$ and involves the standard normal CDF if $\alpha = 1/2, 2$.

The main results of this note are explicit closed form expressions for the CHF for the EVDs of type 2 and type 3. These results are new.

Sections 2 and 3 provide expressions for $\phi_X(t)$ when X has the CDFs Eqs. 1.2 and 1.3, respectively. These expressions involve the Fox's H function and the Wright generalized hypergeometric function. Section 4 discusses how the results of Sections 2 and 3 can be applied in practice.

For integers $m, n, p, q \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that $m \le q, n \le p$, and for $a_i, b_j \in \mathbb{C}$ and $A_i, B_j > 0$, $i = \overline{1, p}$, $j = \overline{1, q}$, the H function, $H^{m,n}_{p,q}(z)$, is defined *via* a Mellin-Barnes integral in the form

$$\begin{aligned} H^{m,n}_{p,q}(z) &\equiv H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right] \\ &= H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right] \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Xi^{m,n}_{p,q}(s) z^{-s} \mathrm{d}s \end{aligned} \tag{1.5}$$

with

$$\Xi_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{k=1}^{n} \Gamma\left(1 - a_k - A_k s\right)}{\prod_{k=n+1}^{p} \Gamma\left(a_k + A_k s\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right)}. \tag{1.6}$$

Here, $z^{-s} = \exp\{-s \ln|z| + \mathrm{i} \arg z\}$, $z \neq 0$, where $\arg z$ is not necessarily the principal value, while the empty product in Eq. 1.6 is taken to be one. The integration path $\mathcal{L}$ is an infinite contour which separates all poles of the gamma functions $\Gamma(b_j + B_j s)$, $j = \overline{1, m}$ to the left and all poles of the gamma functions $\Gamma(1 - a_k - A_k s)$, $k = \overline{1, n}$ to the right of $\mathcal{L}$. Precise details can be found in Mathai and Saxena (1978, Chapter 1), Srivastava et al. (1982, Chapter 1) and Kilbas et al. (2006).

The complex parameter Wright generalized hypergeometric function, ${}_p\Psi_q(\cdot)$, with p numerator and q denominator parameters (Kilbas et al. 2006, Equation (1.9)) is defined by the series

$${}_p\Psi_q\left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(\alpha_j + A_j n\right)}{\prod_{j=1}^{q} \Gamma\left(\beta_j + B_j n\right)} \frac{z^n}{n!} \tag{1.7}$$

for $z \in \mathbb{C}$, where $\alpha_j, \beta_k \in \mathbb{C}$, $A_j, B_k \neq 0$, $j = \overline{1, p}$, $k = \overline{1, q}$ and the series converges for $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$, compare with Mathai and Saxena (1978) and Srivastava et al. (1982). This function was originally introduced by Wright (1935).

If any of the parameters (m, n, p, q) in Eq. 1.5 is zero then the corresponding product in Eq. 1.6 should be taken as one. For example, if $n = 0$ then the second product in the numerator of Eq. 1.6 should be taken as one. If $p = 0$ then the first product in the denominator of Eq. 1.6 should be taken as one. If $m = n = p = q = 0$ then Eq. 1.5 is undefined.

Similarly, if any of the parameters (p, q) in Eq. 1.7 is zero then the corresponding product should be taken as one. For example, if $p = 0$ then the product in the numerator of Eq. 1.7 should be taken as one. If $q = 0$ then the product in the denominator of Eq. 1.7 should be taken as one. If $p = q = 0$ then Eq. 1.7 is undefined.

Hypergeometric functions are included as in-built functions in most mathematical software packages, so the special functions in Eqs. 1.5 and 1.7 can be easily evaluated by the software packages **Maple**, Matlab and *Mathematica* using known procedures.

2 CHF for the EVD of type 2

Define on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ a random variable X having the CDF Eq. 1.2. The corresponding PDF is

$$f_X(x) = \frac{\alpha}{\sigma} \left(\frac{x - \mu}{\sigma} \right)^{-\alpha - 1} \exp\left\{ - \left(\frac{x - \mu}{\sigma} \right)^{-\alpha} \right\}$$

for $x > \mu$. The corresponding CHF is

$$\phi_X(t) = \mathsf{E}\exp\{\mathrm{i}tX\} = \frac{\alpha}{\sigma}\int_\mu^\infty \left(\frac{x-\mu}{\sigma}\right)^{-\alpha-1}\exp\left\{\mathrm{i}tx - \left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right\}\mathrm{d}x$$

$$= \alpha\exp\{\mathrm{i}\mu t\}\int_0^\infty u^{\alpha-1}\exp\left\{\mathrm{i}\sigma t u^{-1} - u^\alpha\right\}\mathrm{d}u. \tag{2.1}$$

Consider the integral

$$Z_\rho^\nu(z) = \int_0^\infty u^{\nu-1}\exp\left\{-u^\rho - z/u\right\}\mathrm{d}u$$

for $\rho \in \mathbb{R}$, $\nu \in \mathbb{C}$ and $z \in \mathbb{C}\backslash\{0\}$. This integral is referred to as the complex parameter Krätzel function, studied in detail by Kilbas et al. (2006). The Mellin-transform of $Z_\rho^\nu(z)$ is (Kilbas et al. 2006, Lemma 3.1):

$$\left(\mathcal{M}Z_\rho^\nu\right)(s) = \int_0^\infty Z_\rho^\nu(z)z^{s-1}\mathrm{d}z = \frac{1}{|\rho|}\Gamma(s)\Gamma\left(\frac{\nu+s}{\rho}\right)$$

for $\rho \neq 0$ and $\nu \in \mathbb{C}$. On the other hand, by the definition Eq. 1.5 of the H function, it follows

$$\left(\mathcal{M}H_{p,q}^{m,n}\right)(s) = \Xi_{p,q}^{m,n}(s).$$

That is, $Z_\rho^\nu(z)$ in the case $\rho > 0$ can be represented as the H function in the following form:

$$Z_\rho^\nu(z) = \frac{1}{\rho}H_{0,2}^{2,0}\left[z\,\middle|\,\begin{matrix} - \\ (0,1),\,(\nu/\rho,\,1/\rho)\end{matrix}\right]$$

for $\rho > 0$ and for all $z \in \mathbb{C}\backslash\{0\}$. Now, comparison with Eq. 2.1 by setting $\nu = \rho = \alpha$ leads us to the following result.

Theorem 2.1 *Let X be a random variable having the* EVD *of* type 2. *Then the* CHF *has the closed form:*

$$\phi_X(t) = \begin{cases}\exp\{\mathrm{i}\mu t\}\,H_{0,2}^{2,0}\left[-\mathrm{i}\sigma t\,\middle|\,\begin{matrix} - \\ (0,1),\,(1,\,1/\alpha)\end{matrix}\right], & t \neq 0,\\ 1, & t = 0\end{cases}$$

for all $\alpha > 0$ and $\alpha \notin \mathbb{Q}_0^+ = \{r/s : r \in \mathbb{N}_0, s \in \mathbb{N}\}$.

Proof It remains to prove the continuity of $\phi_X(t)$ at zero, and to prove that $\phi_X(t) \to 0$ for $|t| \to \infty$. According to Kilbas et al. (2006, Corollary 4.2, Remark 4.1), we have

$$Z_\rho^\nu(z) = \begin{cases} \frac{1}{\rho}\Gamma\left(\frac{\nu}{\rho}\right) + O\left(z^{\min(1,\Re\{\nu\})}\right), & \nu + \rho m \notin \mathbb{N}_0 \text{ for all } m \in \mathbb{N}_0, \\ -\frac{1}{\rho}\ln z + O\left(z \ln z\right), & \nu + \rho m \in \mathbb{N}_0 \text{ for some } m \in \mathbb{N}_0 \end{cases}$$

for $\rho > 0$ and as $z \to 0$. We deduce for all $\alpha > 0$, $\alpha \notin \mathbb{Q}_0^+ = \{r/s : r \in \mathbb{N}_0, s \in \mathbb{N}\}$, the following asymptotic behavior:

$$\phi_X(t) = \alpha \exp\{i\mu t\}\, Z_\alpha^\alpha\left(-i\sigma t\right) = \exp\{i\mu t\}\left(1 + O\left[t^{\min(1,\alpha)}\right]\right)$$

as $t \to 0$ that ensures $\phi_X(0) = 1$.

However, for $\alpha \in \mathbb{Q}_0^+$ positive and rational, the continuity property is not fulfilled. Taking into account Kilbas et al. (2010, Theorem 3.2), we have

$$Z_\rho^\nu(z) = \mathfrak{a} z^{(2\nu-\rho)/(2\rho+2)} \exp\left\{-\mathfrak{b} z^{\rho/(\rho+1)}\right\}\left[1 + O\left(z^{-\rho/(\rho+1)}\right)\right]$$

as $z \to \infty$. Here, the convergence is uniform on $|\arg z| < (\rho+1)\pi/(2\rho) - \epsilon$ for a constant $\epsilon \in (0, (\rho+1)\pi/(2\rho))$ and

$$\mathfrak{a} = \left(\frac{2\pi}{\rho+1}\right)^{1/2}, \quad \mathfrak{b} = (1+\rho)\rho^{-\rho/(\rho+1)}.$$

This result gives

$$|\phi_X(t)| = O\left(|t|^{\alpha/(2\alpha+2)} \exp\left\{-\mathfrak{c}|t|^{\alpha/(\alpha+1)}\right\}\right) \to 0$$

as $|t| \to \infty$, where

$$\mathfrak{c} = (1+\alpha)\left(\frac{\sigma}{\alpha}\right)^{\alpha/(\alpha+1)}.$$

The proof is complete. □

3 CHF for the EVD of type 3

Consider a random variable X defined on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ having the CDF Eq. 1.3. The corresponding PDF is

$$f_X(x) = \frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^\alpha\right\}$$

for $x > \mu$. The corresponding CHF is

$$\phi_X(t) = \mathsf{E}\exp\{\mathrm{i}tX\} = \frac{\alpha}{\sigma}\int_\mu^\infty \left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} \exp\left\{\mathrm{i}tx - \left(\frac{x-\mu}{\sigma}\right)^\alpha\right\}\mathrm{d}x$$
$$= \alpha\exp\left\{\mathrm{i}\mu t\right\}\int_0^\infty x^{\alpha-1}\exp\left\{\mathrm{i}\sigma tx - x^\alpha\right\}\mathrm{d}x.$$

Now, repeating the same procedure as in Section 2, we obtain

$$\phi_X(t) = \exp\left\{\mathrm{i}\mu t\right\}\sum_{m=0}^\infty \Gamma\Big(1+\frac{m}{\alpha}\Big)\frac{(\mathrm{i}\sigma t)^m}{m!}.$$

The convergence of the series is fulfilled for all $\alpha > 1$, so using the definition of the Wright function, we have the following result.

Theorem 3.1 *Let X be a random variable having the* EVD *of* type 3. *Then the* CHF *has the closed form:*

$$\phi_X(t) = \exp\left\{\mathrm{i}\mu t\right\}{}_1\Psi_0\Big[\begin{matrix}(1, 1/\alpha)\\ -\end{matrix}\;;\mathrm{i}t\sigma\Big]$$

for $\alpha > 1$.

The EVD of type 3 for $\alpha = 1$ is the exponential distribution. The CHF for the exponential distribution is well known. It is an open problem to derive the CHF for the EVD of type 3 for $\alpha \in (0, 1)$.

4 Application

We mentioned in Section 1 that one use of the results in Sections 2–3 is to compute the distribution of $X_1 + X_2 + \cdots + X_N$ when X_i are independent and identically distributed. A probability of importance associated with this sum is

$$P = \Pr\left(X_1 + X_2 + \cdots + X_N > u\right), \tag{4.1}$$

where N could be deterministic or stochastic. For example, (4.1) could represent the probability that the total claim amount over some period exceeding a certain threshold (Klugman et al. 2008) or the probability that the accumulated failure of a system exceeding some critical level.

Consider the case N is deterministic. It is of interest to know the distribution of $X_1+X_2+\cdots+X_N = Z$ say. If X_i are independent and identical EV random variables of type 2 then, using Theorem 2.1 and the inversion theorem, we can express the PDF of Z as

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^\infty \exp\left\{(N\mu - z)\,\mathrm{i}t\right\}\left\{H_{0,2}^{2,0}\Big[-\mathrm{i}\sigma t\,\Big|\,\begin{matrix}-\\ (0,1), (1, 1/\alpha)\end{matrix}\Big]\right\}^N dt \tag{4.2}$$

provided $\alpha > 0$ and $\alpha \notin \mathbb{Q}_0^+ = \{r/s : r \in \mathbb{N}_0, s \in \mathbb{N}\}$. Using the inversion theorem of Wendel (1961), we can express the CDF of Z as

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty t^{-1} \mathrm{Im}\left[\exp\{(N\mu - z)\,\mathrm{i}t\} \left\{ H_{0,2}^{2,0}\left[-\mathrm{i}\sigma t \,\middle|\, \begin{matrix} - \\ (0,1), (1, 1/\alpha) \end{matrix} \right] \right\}^N \right] dt \tag{4.3}$$

provided $\alpha > 0$ and $\alpha \notin \mathbb{Q}_0^+ = \{r/s : r \in \mathbb{N}_0, s \in \mathbb{N}\}$, where $\mathrm{Im}(\cdot)$ denotes the imaginary part.

Similarly, if X_i are independent and identical EV random variables of type 3 then, using Theorem 3.1 and the inversion theorem, we can express the PDF and the CDF of Z as

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^\infty \exp\{(N\mu - z)\,\mathrm{i}t\} \left\{ {}_1\Psi_0\left[\begin{matrix} (1, 1/\alpha) \\ - \end{matrix} ; \mathrm{i}t\sigma \right] \right\}^N dt \tag{4.4}$$

and

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty t^{-1} \mathrm{Im}\left[\exp\{(N\mu - z)\,\mathrm{i}t\} \left\{ {}_1\Psi_0\left[\begin{matrix} (1, 1/\alpha) \\ - \end{matrix} ; \mathrm{i}t\sigma \right] \right\}^N \right] dt, \tag{4.5}$$

respectively, provided $\alpha > 1$.

The integrals in Eqs. 4.2–4.5 do not appear to have closed forms. However, they can be easily computed using known routines for Fox's H function and Wright generalized hypergeometric Ψ function.

The probability, P, in Eq. 4.1 follows from Eqs. 4.3 and 4.5. If X_i are independent and identical EV random variables of type 2 then

$$P = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty t^{-1} \mathrm{Im}\left[\exp\{(N\mu - u)\,\mathrm{i}t\} \left\{ H_{0,2}^{2,0}\left[-\mathrm{i}\sigma t \,\middle|\, \begin{matrix} - \\ (0,1), (1, 1/\alpha) \end{matrix} \right] \right\}^N \right] dt. \tag{4.6}$$

If X_i are independent and identical EV random variables of type 3 then

$$P = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty t^{-1} \mathrm{Im}\left[\exp\{(N\mu - u)\,\mathrm{i}t\} \left\{ {}_1\Psi_0\left[\begin{matrix} (1, 1/\alpha) \\ - \end{matrix} ; \mathrm{i}t\sigma \right] \right\}^N \right] dt. \tag{4.7}$$

Note that both Eqs. 4.6 and 4.7 are single integrals of known special functions. These representations are perhaps the simplest means to compute P, compare with Zhang (1999). They circumvent the need for approximations for P. The probability, P, can also be computed in other ways. For example, it can be computed by using the PDF of EV random variables. But P will then be an $(N - 1)$-fold integral, a much more complicated representation than Eqs. 4.6 and 4.7.

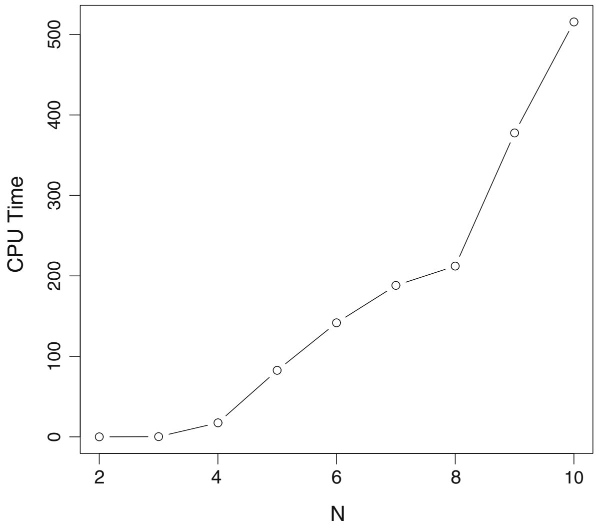

Fig. 1 CPU times taken to compute Eq. 4.6 versus N when $u = 1$, $\mu = 0$, $\sigma = 1$ and $\alpha = \pi$

We now present some numerical results with respect to computing Eqs. 4.6 and 4.7. We take $u = 1$, $\mu = 0$, $\sigma = 1$ and $\alpha = \pi$ in Eq. 4.6. We take $u = 1$, $\mu = 0$, $\sigma = 1$ and $\alpha = 2$ in Eq. 4.7. Figures 1 and 2 show the Central Processing Unit (CPU) time in seconds taken to compute Eqs. 4.6 and 4.7. The figures show how the time varies with respect to N. As expected, the CPU time increases with N. The increase appears sharp. However, it is comforting to note that the CPU times are manageable even for N as large as ten.

The computations for Figures 1 and 2 were performed using $\mathcal{M}athematica$. The accuracy of computations of Eqs. 4.6 and 4.7 is not an issue as $\mathcal{M}athematica$ (like most other algebraic manipulation packages) allows for arbitrary precision.

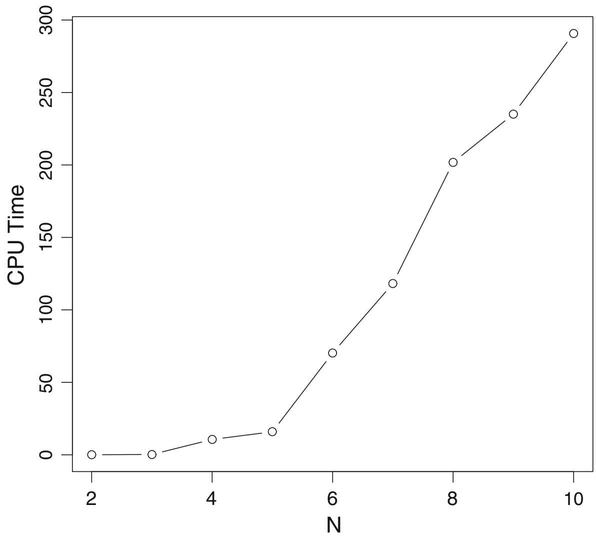

Fig. 2 CPU times taken to compute Eq. 4.7 versus N when $u = 1$, $\mu = 0$, $\sigma = 1$ and $\alpha = 2$

Finally, we like to comment on how the results of this note compare to known results for stable random variables. For stable random variables, the CHF takes an elementary form. The corresponding PDF does not take an elementary form, see Nolan (2012). For EV random variables, the CHFs do not take elementary forms. The corresponding PDFs are elementary. So, computation of probabilities of the form Eq. 4.1 based on CHFs will be easier for stable random variables. On the other hand, estimation and hypothesis testing based on likelihood functions will be more difficult for stable random variables. However, for both stable and EV random variables, it is more convenient to work with CHFs than PDFs for computing probabilities of the form Eq. 4.1.

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