SOLUTIONS TO MATH38181 EXTREME VALUES EXAM

Solutions to Question 1

a) We can write

$$\overline{F}(x,y) = \exp\left\{-(x+y)\left[1 - \frac{2y}{3(x+y)} + \frac{y^2}{3(x+y)^2} + \frac{y^3}{3(x+y)^3}\right]\right\}$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = 1 - 2t/3 + t^2/3 + t^3/3$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since $1 - 2t/3 \geq 0$ for all t and $t^2/3 + t^3/3 \geq 0$ for all t.

Also $A(t) \leq 1$ since

$$\begin{aligned}
& A(t) \leq 1 \\
\Leftrightarrow\ & 1 - 2t/3 + t^2/3 + t^3/3 \leq 1 \\
\Leftrightarrow\ & -2t/3 + t^2/3 + t^3/3 \leq 0 \\
\Leftrightarrow\ & -2/3 + t/3 + t^2/3 \leq 0 \\
\Leftrightarrow\ & -2 + t + t^2 \leq 0 \\
\Leftrightarrow\ & (t+2)(t-1) \leq 0 \\
\Leftrightarrow\ & t - 1 \leq 0.
\end{aligned}$$

Note that

$$\begin{aligned}
& A(t) \geq t \\
\Leftrightarrow\ & 1 - 2t/3 + t^2/3 + t^3/3 \geq t \\
\Leftrightarrow\ & 1 - 5t/3 + t^2/3 + t^3/3 \geq 0 \\
\Leftrightarrow\ & 3 - 5t + t^2 + t^3 \geq 0.
\end{aligned}$$

Let $g(t) = 3 - 5t + t^2 + t^3$. Note $g^{'}(t) = -5 + 2t + 3t^2 = (3t+5)(t-1) \leq 0$ for all t. So, $g(t)$ is a decreasing function with $g(0) = 3$ and $g(1) = 0$. Hence $g(t) \geq t$ for all t.

Note that

$$\begin{aligned}
& A(t) \geq 1 - t \\
\Leftrightarrow\ & 1 - 2t/3 + t^2/3 + t^3/3 \geq 1 - t \\
\Leftrightarrow\ & t/3 + t^2/3 + t^3/3 \geq 0 \\
\Leftrightarrow\ & t + t^2 + t^3 \geq 0.
\end{aligned}$$

$A(\cdot)$ is convex since

$$A^{'}(t) = -2/3 + 2t/3 + t^2$$

and

$$A^{''}(t) = 2/3 + 2t \geq 0$$

for all t.

UNSEEN

b) the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-x - \frac{y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}.$$

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= \exp(-x) - \left(-1 + \frac{y^2}{3(x+y)^2} + \frac{2y^3}{3(x+y)^3}\right) \\ &\cdot \exp\left\{-x - \frac{y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}, \end{aligned}$$

so the conditional cdf if Y given $X = x$ is

$$\begin{aligned} F(y|x) &= 1 - \left(-1 + \frac{y^2}{3(x+y)^2} + \frac{2y^3}{3(x+y)^3}\right) \\ &\cdot \exp\left\{-\frac{y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}. \end{aligned}$$

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= \exp(-y) - \left(-\frac{1}{3} + \frac{2y}{3(x+y)} - \frac{2y^2}{3(x+y)^2} + \frac{2y^3}{3(x+y)^3}\right) \\ &\cdot \exp\left\{-x - \frac{y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}, \end{aligned}$$

so the conditional cdf if X given $Y = y$ is

$$\begin{aligned} F(x|y) &= 1 - \left(-\frac{1}{3} + \frac{2y}{3(x+y)} - \frac{2y^2}{3(x+y)^2} + \frac{2y^3}{3(x+y)^3}\right) \\ &\cdot \exp\left\{-x + \frac{2y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}. \end{aligned}$$

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$f(x,y)=\frac{\partial F(x,y)}{\partial x \partial y}=-\exp\left\{-x-\frac{y}{3}-\frac{y^2}{3(x+y)}-\frac{y^3}{3(x+y)^2}\right\}$$
$$\cdot\left[\left(-1+\frac{y^2}{3(x+y)^2}+\frac{2y^3}{3(x+y)^3}\right)\left(-\frac{1}{3}+\frac{2y}{3(x+y)}-\frac{2y^2}{3(x+y)^2}+\frac{2y^3}{3(x+y)^3}\right)\right.$$
$$\left.+\frac{2y}{3(x+y)^2}-\frac{4y^2}{3(x+y)^3}-\frac{2y^3}{(x+y)^4}\right].$$

UNSEEN

Solutions to Question 2

a) Let X denote the actual stock return. The cdf of X is

$$\begin{aligned}
F_X(x) &= \int_0^\infty \frac{x+\theta}{2\theta} \frac{\lambda}{\theta^2} \exp\left(-\frac{\lambda}{\theta}\right) d\theta \\
&= \frac{\lambda x}{2} \int_0^\infty \frac{1}{\theta^3} \exp\left(-\frac{\lambda}{\theta}\right) d\theta + \frac{\lambda}{2} \int_0^\infty \frac{1}{\theta^2} \exp\left(-\frac{\lambda}{\theta}\right) d\theta \\
&= \frac{x}{2\lambda} \int_0^\infty y \exp\left(-y\right) dy + \frac{1}{2} \int_0^\infty \exp\left(-y\right) dy \\
&= \frac{x}{2\lambda} + \frac{1}{2} \\
&= \frac{x+\lambda}{2\lambda},
\end{aligned}$$

the cdf of the uniform $[-\lambda, \lambda]$ distribution.

UNSEEN

b) The corresponding pdf is $1/(2\lambda)$ for $-\lambda < x < \lambda$.

UNSEEN

c) The corresponding expected value of X is

$$\begin{aligned}
E(X) &= \int_{-\lambda}^{\lambda} x \frac{1}{2\lambda} dx \\
&= \frac{1}{2\lambda} \left[\frac{x^2}{2}\right]_{-\lambda}^{\lambda} \\
&= \frac{1}{2\lambda} \left[\frac{\lambda^2}{2} - \frac{(-\lambda)^2}{2}\right] \\
&= 0.
\end{aligned}$$

UNSEEN

d) The corresponding variance of X is $(2\lambda)^2/12 = \lambda^2/3$.

$$
\begin{aligned}
Var(X) &= E\left(X^2\right) - 0^2 \\
&= \int_{-\lambda}^{\lambda} x^2 \frac{1}{2\lambda} dx \\
&= \frac{1}{2\lambda}\left[\frac{x^3}{3}\right]_{-\lambda}^{\lambda} \\
&= \frac{1}{2\lambda}\left[\frac{\lambda^3}{3} - \frac{(-\lambda)^3}{3}\right] \\
&= \frac{1}{2\lambda}\frac{2\lambda^3}{3} \\
&= \frac{\lambda^2}{3}.
\end{aligned}
$$

UNSEEN

e) If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$
\begin{aligned}
L(\lambda) &= (2\lambda)^{-n} \prod_{i=1}^{n} I\left\{-\lambda < x_i < \lambda\right\} \\
&= (2\lambda)^{-n} I\left\{\max x_i < \lambda, \min x_i > -\lambda\right\} \\
&= (2\lambda)^{-n} I\left\{\lambda > \max x_i, \lambda > -\min x_i\right\} \\
&= (2\lambda)^{-n} I\left\{\lambda > \max\left(\max x_i, -\min x_i\right)\right\}.
\end{aligned}
$$

PLOT THIS AS A FUNCTION OF λ

You will see that the mle of λ is $\max\left(\max x_i, -\min x_i\right)$.

UNSEEN

Solutions to Question 3

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \le x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I \quad &: \quad \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II \quad &: \quad \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \ge 0 \end{cases} \\
& \quad\ \text{for some } \alpha > 0; \\
III \quad &: \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \ge 0 \end{cases} \\
& \quad\ \text{for some } \alpha > 0.
\end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I \quad &: \quad \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II \quad &: \quad w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III \quad &: \quad w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

UP TO THIS BOOK WORK

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$
\begin{aligned}
\lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \lim_{t \to w(G)} \frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)^{\theta}\right]^2\right\}^{\alpha}}{1 - \left\{1 - \left[1 - G\left(t\right)^{\theta}\right]^2\right\}^{\alpha}} \\
&= \lim_{t \to w(G)} \frac{\left[1 - G\left(t + xh(t)\right)^{\theta}\right]^2}{\left[1 - G\left(t\right)^{\theta}\right]^2} \\
&= \left[\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)^{\theta}}{1 - G\left(t\right)^{\theta}}\right]^2 \\
&= \left[\lim_{t \to w(G)} \frac{1 - \left[1 - \left[1 - G\left(t + xh(t)\right)\right]\right]^{\theta}}{1 - \left[1 - \left[1 - G\left(t\right)\right]\right]^{\theta}}\right]^2 \\
&= \left[\lim_{t \to w(G)} \frac{1 - \left[1 - \theta\left[1 - G\left(t + xh(t)\right)\right]\right]}{1 - \left[1 - \theta\left[1 - G\left(t\right)\right]\right]}\right]^2 \\
&= \left[\lim_{t \to w(G)} \frac{\theta\left[1 - G\left(t + xh(t)\right)\right]}{\theta\left[1 - G\left(t\right)\right]}\right]^2 \\
&= \left[\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)}\right]^2 \\
&= e^{-2x}
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-2x)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$\begin{aligned}
\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty} \frac{1-\left\{1-\left[1-G(tx)^{\theta}\right]^{2}\right\}^{\alpha}}{1-\left\{1-\left[1-G(t)^{\theta}\right]^{2}\right\}^{\alpha}} \\
&= \lim_{t\to\infty} \frac{\left[1-G(tx)^{\theta}\right]^{2}}{\left[1-G(t)^{\theta}\right]^{2}} \\
&= \left[\lim_{t\to\infty} \frac{1-G(tx)^{\theta}}{1-G(t)^{\theta}}\right]^{2} \\
&= \left[\lim_{t\to\infty} \frac{1-[1-[1-G(tx)]]^{\theta}}{1-[1-[1-G(t)]]^{\theta}}\right]^{2} \\
&= \left[\lim_{t\to\infty} \frac{1-[1-\theta\,[1-G(tx)]]}{1-[1-\theta\,[1-G(t)]]}\right]^{2} \\
&= \left[\lim_{t\to\infty} \frac{\theta\,[1-G(tx)]}{\theta\,[1-G(t)]}\right]^{2} \\
&= \left[\lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)}\right]^{2} \\
&= x^{-2\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n-b_n}{a_n} \le x\right) = \exp\left(-x^{-2\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\beta}$$

for every $x > 0$. But

$$\begin{aligned}
\lim_{t\to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} &= \lim_{t\to 0} \frac{1 - \left\{1 - \left[1 - G\left(w(G) - tx\right)^{\theta}\right]^2\right\}^{\alpha}}{1 - \left\{1 - \left[1 - G\left(w(G) - t\right)^{\theta}\right]^2\right\}^{\alpha}} \\
&= \lim_{t\to 0} \frac{\left[1 - G\left(w(G) - tx\right)^{\theta}\right]^2}{\left[1 - G\left(w(G) - t\right)^{\theta}\right]^2} \\
&= \left[\lim_{t\to 0} \frac{1 - G\left(w(G) - tx\right)^{\theta}}{1 - G\left(w(G) - t\right)^{\theta}}\right]^2 \\
&= \left[\lim_{t\to 0} \frac{1 - \left[1 - \left[1 - G\left(w(G) - tx\right)\right]\right]^{\theta}}{1 - \left[1 - \left[1 - G\left(w(G) - t\right)\right]\right]^{\theta}}\right]^2 \\
&= \left[\lim_{t\to 0} \frac{1 - \left[1 - \theta\left[1 - G\left(w(G) - tx\right)\right]\right]}{1 - \left[1 - \theta\left[1 - G\left(w(G) - t\right)\right]\right]}\right]^2 \\
&= \left[\lim_{t\to 0} \frac{\theta\left[1 - G\left(w(G) - tx\right)\right]}{\theta\left[1 - G\left(w(G) - t\right)\right]}\right]^2 \\
&= \left[\lim_{t\to \infty} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)}\right]^2 \\
&= x^{2\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{2\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

UNSEEN

Solutions to Question 4

a) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t\uparrow\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\uparrow\infty} \frac{\left[1+(tx)^c\right]^{-k}}{\left[1+t^c\right]^{-k}} \\
&= \lim_{t\uparrow\infty} \left[\frac{1+(tx)^c}{1+t^c}\right]^{-k} \\
&= x^{-ck}.
\end{aligned}$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

UNSEEN

b) Note that $w(F) = 1$. Then

$$\begin{aligned}
\lim_{t\to 0} \frac{1-F(1-tx)}{1-F(1-t)} &= \lim_{t\to 0} \frac{\left[1-(1-tx)^b\right]^a}{\left[1-(1-t)^b\right]^a} \\
&= \lim_{t\to 0} \left[\frac{1-(1-tx)^b}{1-(1-t)^b}\right]^a \\
&= \lim_{t\to 0} \left[\frac{1-(1-btx)}{1-(1-bt)}\right]^a \\
&= \lim_{t\to 0} \left[\frac{btx}{bt}\right]^a \\
&= x^a.
\end{aligned}$$

So, F belongs to the Weibull domain of attraction.

UNSEEN

c) For the Poisson distribution,

$$\frac{\Pr(X=k)}{1-F(k-1)} = \frac{\lambda^k/k!}{\sum_{j=k}^{\infty}\lambda^j/j!} = \frac{1}{1+\sum_{j=k+1}^{\infty} k!\lambda^{j-k}/j!}.$$

The term in the denominator can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \leq \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1-\lambda/k}$$

(when $k > \lambda$) and the bound tends to 0 as $k \to \infty$ and so it follows that $p(k)/(1-F(k-1)) \to 1$. Hence, there can be no non-degenerate limit.

CLASS EXERCISE

d) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \Phi\left(t + xg(t)\right)}{1 - \Phi(t)} \\
&= \lim_{t \to \infty} \frac{\phi\left(t + xg(t)\right)}{\phi(t)} \left(1 + xg^{'}(t)\right) \\
&= \lim_{t \to \infty} \exp\left\{-\frac{1}{2}\left[2txg(t) + x^2 g^2(t)\right]\right\} \left(1 + xg^{'}(t)\right) \\
&= \lim_{t \to \infty} \exp\left\{-x - \frac{x^2}{2t^2}\right\} \left(1 - \frac{x}{t^2}\right) \\
&= \exp(-x)
\end{aligned}$$

if $g(t) = 1/t$. So, F belongs to the Gumbel domain of attraction.

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \exp\left\{-\exp\left[-t - xg(t)\right]\right\}}{1 - \exp\left\{-\exp(-t)\right\}} \\
&= \lim_{t \to \infty} \frac{1 - \left\{1 - \exp\left[-t - xg(t)\right]\right\}}{1 - \left\{1 - \exp(-t)\right\}} \\
&= \lim_{t \to \infty} \frac{\exp\left[-t - xg(t)\right]}{\exp(-t)} \\
&= \lim_{t \to \infty} \exp\left[-xg(t)\right] \\
&= \exp(-x)
\end{aligned}$$

if $g(t) = 1$. So, F belongs to the Gumbel domain of attraction.

UNSEEN

Solutions to Question 5

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

UP TO THIS BOOK WORK

(b) (i) If $x > 0$ then

$$\begin{aligned}
F(x) &= \frac{1}{2\lambda} \int_{-\infty}^{x} \exp\left(-\frac{\mid y \mid}{\lambda}\right) dy \\
&= 1 - \frac{1}{2\lambda} \int_{x}^{\infty} \exp\left(-\frac{\mid y \mid}{\lambda}\right) dy \\
&= 1 - \frac{1}{2\lambda} \int_{-\infty}^{x} \exp\left(-\frac{y}{\lambda}\right) dy \\
&= 1 - \frac{1}{2} \left[- \exp\left(-\frac{y}{\lambda}\right)\right]_{-\infty}^{x} \\
&= 1 - \frac{1}{2} \exp\left(-\frac{x}{\lambda}\right).
\end{aligned}$$

If $x < 0$ then

$$\begin{aligned}
F(x) &= \frac{1}{2\lambda} \int_{-\infty}^{x} \exp\left(-\frac{\mid y \mid}{\lambda}\right) dy \\
&= \frac{1}{2\lambda} \int_{-\infty}^{x} \exp\left(\frac{y}{\lambda}\right) dy \\
&= \frac{1}{2} \left[\exp\left(\frac{y}{\lambda}\right)\right]_{-\infty}^{x} \\
&= \frac{1}{2} \exp\left(\frac{x}{\lambda}\right).
\end{aligned}$$

UNSEEN

(b) (ii) Inverting

$$F(x) = 1 - \frac{1}{2} \exp\left(-\frac{x}{\lambda}\right) = p,$$

we obtain $\mathrm{VaR}_p(X) = \lambda \log\left[2(1-p)\right]$ for $p \geq 1/2$. Inverting

$$F(x) = \frac{1}{2} \exp\left(\frac{x}{\lambda}\right) = p,$$

we obtain $\mathrm{VaR}_p(X) = \lambda \log [2p]$ for $p < 1/2$.

UNSEEN

(b) (iii) Since

$$\begin{aligned} \int_0^p \log t dt &= [t \log t]_0^p - \int_0^p 1 \cdot dt \\ &= p (\log p - 1) \end{aligned}$$

and

$$\begin{aligned} \int_{1/2}^p \log(1-t) dt &= [t \log(1-t)]_{1/2}^p + \int_{1/2}^p \frac{t}{1-t} dt \\ &= p \log(1-p) - \frac{1}{2} \log \frac{1}{2} + \int_{1/2}^p \frac{t}{1-t} dt \\ &= p \log(1-p) - \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} - p + \int_{1/2}^p \frac{1}{1-t} dt \\ &= p \log(1-p) - \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} - p + [-\log(1-t)]_{1/2}^p \\ &= p \log(1-p) - \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} - p - \log(1-p) - \log 2 \\ &= p \log(1-p) - \frac{1}{2} \log 2 + \frac{1}{2} - p - \log(1-p), \end{aligned}$$

we obtain

$$\mathrm{ES}_(X) = \lambda \log 2 + \lambda (\log p - 1)$$

for $p < 1/2$ and

$$\mathrm{ES}_p(X) = \frac{1}{p} [p \lambda \log 2 + \lambda p \log(1-p) - \lambda \log 2 - \lambda p - \lambda \log(1-p)]$$

for $p \geq 1/2$.

UNSEEN

c) (i) The likelihood and log-likelihood functions of λ are

$$L(\lambda) = \frac{1}{(2\lambda)^n} \exp \left[-\frac{1}{\lambda} \sum_{i=1}^n |x_i| \right]$$

and

$$\log L(\lambda) = -n \log(2\lambda) - \frac{1}{\lambda} \sum_{i=1}^n |x_i| .$$

UNSEEN

c) (ii)

$$\frac{d \log L}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^{n} |x_i|$$

and

$$\frac{d^2 \log L}{d\lambda^2} = \frac{n}{\lambda^2} - \frac{2}{\lambda^3} \sum_{i=1}^{n} |x_i| \, .$$

The root of $\frac{d \log L}{d\lambda} = 0$ is $\displaystyle\frac{1}{n} \sum_{i=1}^{n} |x_i| = \widehat{\lambda}$ say. The value of $\frac{d^2 \log L}{d\lambda^2}$ at $\lambda = \widehat{\lambda}$ is negative, so $\widehat{\lambda}$ is an mle.

UNSEEN

c) (iii) The MLE of VaR is $\widehat{\mathrm{VaR}}_p(X) = \widehat{\lambda} \log \left[2(1-p) \right]$ for $p \geq 1/2$ and $\widehat{\mathrm{VaR}}_p(X) = \widehat{\lambda} \log \left[2p \right]$ for $p < 1/2$.

UNSEEN

The MLE of ES is

$$\widehat{\mathrm{ES}}_(X) = \widehat{\lambda} \log 2 + \widehat{\lambda} \left(\log p - 1 \right)$$

for $p < 1/2$ and

$$\widehat{\mathrm{ES}}_p(X) = \frac{1}{p} \left[p\widehat{\lambda} \log 2 + \widehat{\lambda} p \log (1-p) - \widehat{\lambda} \log 2 - \widehat{\lambda} p - \widehat{\lambda} \log (1-p) \right]$$

for $p \geq 1/2$.

UNSEEN

c (iv) This follows since $\widehat{\lambda}$ is unbiased for λ, i.e.,

$$\begin{aligned}
E \left(\frac{1}{n} \sum_{i=1}^{n} |x_i| \right) &= \frac{1}{n} \sum_{i=1}^{n} E \left(|x_i| \right) \\
&= \frac{1}{2n\lambda} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mid x \mid \exp \left(-\frac{\mid x \mid}{\lambda} \right) dx \\
&= \frac{1}{\lambda} \int_{0}^{\infty} x \exp \left(-\frac{x}{\lambda} \right) dx \\
&= \lambda \int_{0}^{\infty} y \exp \left(-y \right) dy \\
&= \lambda \Gamma(2) \\
&= \lambda.
\end{aligned}$$

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Solutions to Question 6

a) The cdf of Y is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\min\left(X_1, \ldots, X_\alpha\right) \leq y\right) \\
&= 1 - \Pr\left(\min\left(X_1, \ldots, X_\alpha\right) > y\right) \\
&= 1 - \Pr\left(X_1 > y, \ldots, X_\alpha > y\right) \\
&= 1 - \Pr\left(X_1 > y\right) \cdots \Pr\left(X_\alpha > y\right) \\
&= 1 - \left(\frac{K}{y}\right)^a \cdots \left(\frac{K}{y}\right)^a \\
&= 1 - \left(\frac{K}{y}\right)^{a\alpha},
\end{aligned}$$

a Pareto cdf with parameters K and $a\lambda$.

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b) The corresponding pdf is

$$f_Y(y) = a\alpha \frac{K^{a\alpha}}{y^{a\alpha+1}}$$

for $y \geq K$.

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c) The nth moment of Y can be calculated as

$$\begin{aligned}
E\left(Y^n\right) &= a\alpha K^{a\alpha} \int_K^\infty y^{n-a\alpha-1} dy \\
&= a\alpha K^{a\alpha} \left[\frac{y^{n-a\alpha}}{n - a\alpha}\right]_K^\infty \\
&= a\alpha K^{a\alpha} \left[0 - \frac{K^{n-a\alpha}}{n - a\alpha}\right] \\
&= -a\alpha \frac{K^n}{n - a\alpha}
\end{aligned}$$

provided that $a\alpha > n$. So,

$$E\left(Y\right) = \frac{a\alpha K}{a\alpha - 1}$$

and

$$Var\left(Y\right) = \frac{a\alpha K^2}{a\alpha - 2} - \frac{a^2\alpha^2 K^2}{(a\alpha - 1)^2}$$

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d) Setting

$$1 - \left(\frac{K}{y} \right)^{a\alpha} = p$$

gives

$$\mathrm{VaR}_p(Y) = K \left(1 - p \right)^{-1/(a\alpha)}.$$

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e) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= \frac{K}{p} \int_0^p (1 - v)^{-1/(a\alpha)} \, dv \\
&= \frac{K}{p \left[1 - 1/(a\alpha) \right]} \left[- (1 - v)^{1 - 1/(a\alpha)} \right]_0^p \\
&= \frac{K}{p \left[1 - 1/(a\alpha) \right]} \left[1 - (1 - p)^{1 - 1/(a\alpha)} \right].
\end{aligned}$$

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f) The likelihood and log likelihood functions are

$$L(a, K) = a^n \alpha^n K^{na\alpha} \prod_{i=1}^n \left[y_i^{-a\alpha - 1} I \left\{ y_i \geq K \right\} \right] = a^n \alpha^n K^{na\alpha} \left(\prod_{i=1}^n y_i \right)^{-a\alpha - 1} I \left\{ \min y_i \geq K \right\}$$

and

$$\log L(a, K) = n \log a + n \log \alpha + na\alpha \log K - (a\alpha + 1) \sum_{i=1}^n \log y_i + \log I \left\{ \min y_i \geq K \right\}.$$

Note that L is an increasing function of K over $(0, \min y_i]$. So, the mle of K is $\min y_i$. The partial derivative of $\log L$ with respect to a is

$$\frac{\partial \log L(a, K)}{\partial a} = \frac{n}{a} + n\alpha \log K - \alpha \sum_{i=1}^n \log y_i$$

The solution of $\frac{\partial \log L(a,K)}{\partial a} = 0$ for a is

$$\widehat{a} = - \left[\alpha \log K - \frac{\alpha}{n} \sum_{i=1}^n \log y_i \right]^{-1},$$

the mle of a.

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