Proof of $t_\nu \to N(0,1)$ as $\nu \to \infty$

As $\nu \to \infty$,

$$
\begin{aligned}
\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \quad &\sim \quad \frac{\sqrt{2\pi\frac{\nu-1}{2}}\left(\frac{\nu-1}{2}\right)^{\frac{\nu-1}{2}} e^{-\frac{\nu-1}{2}}}{\sqrt{\nu\pi}\sqrt{2\pi\frac{\nu-2}{2}}\left(\frac{\nu-2}{2}\right)^{\frac{\nu-2}{2}} e^{-\frac{\nu-2}{2}}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \\
&\qquad \left[\text{using } \Gamma(n+1) \sim \sqrt{2\pi n}\, n^n e^{-n}\right] \\
&\sim \quad \frac{1}{\sqrt{\nu\pi}}\sqrt{\frac{\nu-1}{\nu-2}}\,\frac{\left(\frac{\nu-1}{2}\right)^{\frac{\nu}{2}}\left(\frac{\nu-1}{2}\right)^{-\frac{1}{2}} e^{-\frac{\nu-1}{2}}}{\left(\frac{\nu-2}{2}\right)^{\frac{\nu}{2}}\left(\frac{\nu-2}{2}\right)^{-1} e^{-\frac{\nu-2}{2}}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \\
&\sim \quad \frac{1}{\sqrt{2\pi}}\left(\frac{\nu-1}{\nu-2}\right)^{\frac{\nu}{2}} \frac{\nu-2}{\sqrt{\nu}\sqrt{\nu-1}} e^{-\frac{1}{2}}\left[\left(1+\frac{x^2}{\nu}\right)^{\nu}\right]^{-\frac{1+\nu}{2\nu}} \\
&\sim \quad \frac{1}{\sqrt{2\pi}}\left[\left(1+\frac{1}{\nu-2}\right)^{\nu-2}\right]^{\frac{\nu}{2(\nu-2)}} e^{-\frac{1}{2}}\left[\left(1+\frac{x^2}{\nu}\right)^{\nu}\right]^{-\frac{1+\nu}{2\nu}} \\
&\sim \quad \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}} e^{-\frac{1}{2}}\left[\left(1+\frac{x^2}{\nu}\right)^{\nu}\right]^{-\frac{1+\nu}{2\nu}} \\
&\qquad \left[\text{using } \left(1+\frac{y}{n}\right)^n \to e^y \text{ as } n \to \infty\right] \\
&\sim \quad \frac{1}{\sqrt{2\pi}}\left[\left(1+\frac{x^2}{\nu}\right)^{\nu}\right]^{-\frac{1+\nu}{2\nu}} \\
&\to \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
&\qquad \left[\text{using } \left(1+\frac{y}{n}\right)^n \to e^y \text{ as } n \to \infty\right].
\end{aligned}
$$

The proof is complete.