A Student's t random variable say X with $a > 0$ degree of freedom has the probability density function (pdf)

$$f(x) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}$$

for $-\infty < x < +\infty$, where $\Gamma(\cdot)$ denotes the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt.$$

Mean

The mean of X can be written as

$$E(X) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{+\infty} x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx.$$

Note that

$$x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}$$

is an odd function. An odd function say $g(\cdot)$ is a function for which $g(x) = -g(-x)$ for all x. One property of an odd function is that

$$\int_{-\infty}^{+\infty} g(x) dx = 0.$$

Hence,

$$E(X) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{+\infty} x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx = 0.$$

Variance

The variance of X can be written as

$$Var(X) = E\left(X^2\right) - \left[E\left(X\right)\right]^2 = E\left(X^2\right) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{+\infty} x^2 \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx.$$

Note that

$$x^2 \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}$$

is an even function. An even function say $g(\cdot)$ is a function for which $g(x) = g(-x)$ for all x. One property of an even function is that

$$\int_{-\infty}^{+\infty} g(x) dx = 2 \int_0^{+\infty} g(x) dx.$$

Hence,

$$Var(X) = \frac{2\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \int_0^{+\infty} x^2 \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx. \tag{1}$$

Now substitute $y = 1/\left(1 + \frac{x^2}{a}\right)$. Then $x = \sqrt{a}\sqrt{\frac{1-y}{y}}$ and $\frac{dx}{dy} = -\frac{\sqrt{a}}{2}(1-y)^{-\frac{1}{2}}y^{-\frac{3}{2}}$. Under this substitution, (1) can be rewritten as

$$Var(X) = -\frac{a\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)} \int_1^0 y^{\frac{a}{2}-2}\sqrt{1-y}dy = \frac{a\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)} \int_0^1 y^{\frac{a}{2}-2}\sqrt{1-y}dy. \tag{2}$$

Using the fact that a beta function is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,$$

(2) can be reduced to

$$Var(X) = \frac{a\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)} B\left(\frac{a}{2} - 1, \frac{3}{2}\right). \tag{3}$$

Using the property that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

(3) can be further reduced to

$$\begin{aligned}
Var(X) &= \frac{a\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)} \frac{\Gamma\left(\frac{a}{2}-1\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)} \\
&= \frac{a\Gamma\left(\frac{a}{2}-1\right)\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)} \\
&= \frac{a\Gamma\left(\frac{a}{2}-1\right)\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}\left(\frac{a}{2}-1\right)\Gamma\left(\frac{a}{2}-1\right)} \qquad \text{using the fact that } \Gamma(x+1) = x\Gamma(x) \\
&= \frac{a\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}\left(\frac{a}{2}-1\right)} \\
&= \frac{a\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}\left(\frac{a}{2}-1\right)} \qquad \text{using the fact that } \Gamma(x+1) = x\Gamma(x) \\
&= \frac{a\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}\left(\frac{a}{2}-1\right)} \qquad \text{using the fact that } \Gamma(1/2) = \sqrt{\pi} \\
&= \frac{a}{a-2}.
\end{aligned}$$