# MATH4/68181: Extreme values and financial risk
# Semester 1
# Solutions to problem sheet for Week 9

1. for the independence copula defined by $C(u_1, u_2) = u_1 u_2$, we have

$$C(u, 0) = u \cdot 0 = 0,$$

$$C(0, u) = 0 \cdot u = 0,$$

$$C(1, u) = 1 \cdot u = u,$$

$$C(u, 1) = u \cdot 1 = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 \geq 0,$$

so $C$ is a valid copula.

2. for the copula defined by $C(u_1, u_2) = \min(u_1, u_2)$, we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \min(1, u) = u,$$

$$C(u, 1) = \min(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 \leq u_2, \\ 0, & \text{if } u_1 > u_2, \end{cases}$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 \leq u_2, \\ 1, & \text{if } u_1 > u_2, \end{cases}$$

so $C$ is a valid copula.

3. for the copula defined by $C(u_1, u_2) = u_1 u_2 \exp[-\theta \log u_1 \log u_2]$, we have

$$C(u, 0) = u \cdot 0 \cdot \exp[-\theta \log 0 \log u] = 0,$$

$$C(0, u) = 0 \cdot u \exp[-\theta \log u \log 0] = 0,$$

$$C(1, u) = 1 \cdot u \exp[-\theta \log 1 \log u] = u,$$

$$C(u, 1) = u \cdot 1 \exp[-\theta \log u \log 1] = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 (1 - \theta \log u_2) \exp[-\theta \log u_1 \log u_2] \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 (1 - \theta \log u_1) \exp[-\theta \log u_1 \log u_2] \geq 0,$$

so $C$ is a valid copula.

4. for the Farlie-Gumbel-Morgenstern copula defined by

$$C(u_1, u_2) = u_1 u_2 [1 + \phi (1 - u_1)(1 - u_2)],$$

we have

$$C(u, 0) = u \cdot 0 [1 + \phi (1 - u)(1 - 0)] = 0,$$

$$C(0, u) = 0 \cdot u [1 + \phi (1 - 0)(1 - u)] = 0,$$

$$C(u, 1) = u \cdot 1 [1 + \phi (1 - u)(1 - 1)] = u,$$

$$C(1, u) = 1 \cdot u [1 + \phi (1 - 1)(1 - u)] = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= u_2 [1 + \phi (1 - u_1)(1 - u_2)] + u_1 u_2 [-\phi (1 - u_2)] \\ &= u_2 [1 + \phi (1 - 2u_1)(1 - u_2)] \\ &\geq 0 \end{aligned}$$

(since $1 \geq (1 - 2u_1)(1 - u_2) \geq -1$ for all $u_1$, $u_2$) and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= u_1 [1 + \phi (1 - u_1)(1 - u_2)] + u_1 u_2 [-\phi (1 - u_1)] \\ &= u_1 [1 + \phi (1 - u_1)(1 - 2u_2)] \\ &\geq 0 \end{aligned}$$

(since $1 \geq (1 - u_1)(1 - 2u_2) \geq -1$ for all $u_1$, $u_2$), so $C$ is a valid copula.

5. for the Burr copula defined by $C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right]^{-\alpha}$, we have

$$C(u, 0) = u + 0 - 1 + \left[(1-u)^{-1/\alpha} + (1-0)^{-1/\alpha} - 1\right]^{-\alpha} = 0,$$

$$C(0, u) = 0 + u - 1 + \left[(1-0)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1\right]^{-\alpha} = 0,$$

$$\begin{aligned} C(u, 1) &= u + 1 - 1 + \left[(1-u)^{-1/\alpha} + (1-1)^{-1/\alpha} - 1\right]^{-\alpha} \\ &= u + 0 + \left[(1-u)^{-1/\alpha} + (0)^{-1/\alpha} - 1\right]^{-\alpha} \\ &= u + \left[(1-u)^{-1/\alpha} + \infty - 1\right]^{-\alpha} \\ &= u + \infty^{-\alpha} \\ &= u + 0 \\ &= u \end{aligned}$$

$$\begin{aligned} C(1, u) &= 1 + u - 1 + \left[(1-1)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1\right]^{-\alpha} \\ &= u + \left[(0)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1\right]^{-\alpha} \\ &= u + \left[\infty + (1-u)^{-1/\alpha} - 1\right]^{-\alpha} \\ &= u + \infty^{-\alpha} \\ &= u + 0 \\ &= u, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= 1 - (1-u_1)^{-1/\alpha - 1} \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right]^{-\alpha - 1} \\ &= 1 - \left[\frac{(1-u_1)^{-1/\alpha}}{(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1}\right]^{\alpha + 1} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= 1 - (1-u_2)^{-1/\alpha - 1} \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right]^{-\alpha - 1} \\ &= 1 - \left[\frac{(1-u_2)^{-1/\alpha}}{(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1}\right]^{\alpha + 1} \geq 0, \end{aligned}$$

so $C$ is a valid copula.

6. for Marshall and Olkin's copula defined by

$$C(u_1, u_2) = \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^\alpha \geq u_2^\beta, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^\alpha < u_2^\beta, \end{cases}$$

we have

$$C(u,0) = u^{1-\alpha} \cdot 0 = 0,$$

$$C(0,u) = 0 \cdot u^{1-\beta} = 0,$$

$$C(u,1) = u \cdot 1 = u,$$

$$C(1,u) = 1 \cdot u = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1,u_2) &= \begin{cases} (1-\alpha) u_1^{-\alpha} u_2, & \text{if } u_1^\alpha \geq u_2^\beta, \\ \\ u_2^{1-\beta}, & \text{if } u_1^\alpha < u_2^\beta, \end{cases} \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1,u_2) &= \begin{cases} u_1^{1-\alpha}, & \text{if } u_1^\alpha \geq u_2^\beta, \\ \\ (1-\beta) u_1 u_2^{-\beta}, & \text{if } u_1^\alpha < u_2^\beta, \end{cases} \\ &\geq 0, \end{aligned}$$

so $C$ is a valid copula.