MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet 13

1) Consider the ARCH (q) model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2.$$

Then,

$$\begin{aligned} E\left(e_t\right) &= E\left(\sigma_t Z_t\right) \\ &= E\left[E\left(\sigma_t Z_t \mid \sigma_t\right)\right] \\ &= E\left[\sigma_t E\left(Z_t \mid \sigma_t\right)\right] \\ &= E\left[\sigma_t \cdot 0\right] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E\left(e_t^2\right) &= E\left(\sigma_t^2 Z_t^2\right) \\ &= E\left[E\left(\sigma_t^2 Z_t^2 \mid \sigma_t\right)\right] \\ &= E\left[\sigma_t^2 E\left(Z_t^2 \mid \sigma_t\right)\right] \\ &= E\left[\sigma_t^2\right]. \end{aligned}$$

So, the mean is zero and the variance is $E\left[\sigma_t^2\right]$. If stationarity holds

$$\begin{aligned} & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left(e_{t-1}^2\right) + \cdots + \alpha_q E\left(e_{t-q}^2\right) \\ \Leftrightarrow\ & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[E\left(e_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \cdots + \alpha_q E\left[E\left(e_{t-q}^2 \mid \sigma_{t-q}\right)\right] \\ \Leftrightarrow\ & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2 E\left(Z_{t-q}^2 \mid \sigma_{t-q}\right)\right] \\ \Leftrightarrow\ & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2 \cdot 1\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2 \cdot 1\right] \\ \Leftrightarrow\ & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2\right] \\ \Leftrightarrow\ & \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2 \\ \Leftrightarrow\ & \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}. \end{aligned}$$

2) Consider the GARCH (p, q) model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2.$$

Then, as in problem 1,

$$E(e_t) = 0$$

and

$$E\left(e_t^2\right) = E\left[\sigma_t^2\right].$$

So, the mean is zero and the variance is $E\left[\sigma_t^2\right]$. If stationarity holds

$$\begin{aligned}
& E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left(e_{t-1}^2\right) + \cdots + \alpha_q E\left(e_{t-q}^2\right) \\
& \quad +\beta_1 E\left(\sigma_{t-1}^2\right) + \cdots + \beta_p E\left(\sigma_{t-p}^2\right) \\
\Leftrightarrow \quad & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[E\left(e_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \cdots + \alpha_q E\left[E\left(e_{t-q}^2 \mid \sigma_{t-q}\right)\right] \\
& \quad +\beta_1 E\left(\sigma_{t-1}^2\right) + \cdots + \beta_p E\left(\sigma_{t-p}^2\right) \\
\Leftrightarrow \quad & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2 E\left(Z_{t-q}^2 \mid \sigma_{t-q}\right)\right] \\
& \quad +\beta_1 E\left(\sigma_{t-1}^2\right) + \cdots + \beta_p E\left(\sigma_{t-p}^2\right) \\
\Leftrightarrow \quad & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2 \cdot 1\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2 \cdot 1\right] + \beta_1 E\left(\sigma_{t-1}^2\right) + \cdots + \beta_p E\left(\sigma_{t-p}^2\right) \\
\Leftrightarrow \quad & E\left(\sigma_t^2\right) = \alpha_0 + \alpha_1 E\left[\sigma_{t-1}^2\right] + \cdots + \alpha_q E\left[\sigma_{t-q}^2\right] + \beta_1 E\left(\sigma_{t-1}^2\right) + \cdots + \beta_p E\left(\sigma_{t-p}^2\right) \\
\Leftrightarrow \quad & \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2 + \beta_1 \sigma^2 + \cdots + \beta_p \sigma^2 \\
\Leftrightarrow \quad & \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q - \beta_1 - \cdots - \beta_q}.
\end{aligned}$$

3) Consider the NGARCH model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \omega + \alpha\left(e_{t-1} - \theta\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2.$$

Then, as in problem 1,

$$E(e_t) = 0$$

and

$$E\left(e_t^2\right) = E\left[\sigma_t^2\right].$$

So, the mean is zero and the variance is $E\left[\sigma_t^2\right]$. If stationarity holds

$$\begin{aligned}
& E\left(\sigma_t^2\right) = \omega + \alpha E\left[\left(e_{t-1} - \theta\sigma_{t-1}\right)^2\right] + \beta E\left(\sigma_{t-1}^2\right) \\
\Leftrightarrow \quad & E\left(\sigma_t^2\right) = \omega + \alpha E\left(e_{t-1}^2\right) - 2\alpha\theta E\left(e_{t-1}\sigma_{t-1}\right) + \alpha\theta^2 E\left(\sigma_{t-1}^2\right) + \beta E\left(\sigma_{t-1}^2\right) \\
\Leftrightarrow \quad & \sigma^2 = \omega + \alpha E\left[E\left(e_{t-1}^2 \mid \sigma_{t-1}\right)\right] - 2\alpha\theta E\left[E\left(e_{t-1}\sigma_{t-1} \mid \sigma_{t-1}\right)\right] + \alpha\theta^2\sigma^2 + \beta\sigma^2
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad \sigma^2 = \omega + \alpha E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \mid \sigma_{t-1}\right)\right] - 2\alpha\theta E\left[\sigma_{t-1}^2 E\left(Z_{t-1} \mid \sigma_{t-1}\right)\right] + \alpha\theta^2\sigma^2 + \beta\sigma^2 \\
&\Leftrightarrow \quad \sigma^2 = \omega + \alpha E\left[\sigma_{t-1}^2 \cdot 1\right] - 2\alpha\theta E\left[\sigma_{t-1}^2 \cdot 0\right] + \alpha\theta^2\sigma^2 + \beta\sigma^2 \\
&\Leftrightarrow \quad \sigma^2 = \omega + \alpha E\left[\sigma_{t-1}^2\right] + \alpha\theta^2\sigma^2 + \beta\sigma^2 \\
&\Leftrightarrow \quad \sigma^2 = \omega + \alpha\sigma^2 + \alpha\theta^2\sigma^2 + \beta\sigma^2 \\
&\Leftrightarrow \quad \sigma^2 = \frac{\omega}{1 - \alpha - \alpha\theta^2 - \beta}.
\end{aligned}
$$

4) Consider the QGARCH model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = K + \alpha e_{t-1}^2 + \beta\sigma_{t-1}^2 + \phi e_{t-1}.$$

Then, as in problem 1,

$$E\left(e_t\right) = 0$$

and

$$E\left(e_t^2\right) = E\left[\sigma_t^2\right].$$

If stationarity holds

$$
\begin{aligned}
&E\left(\sigma_t^2\right) = K + \alpha E\left(e_{t-1}^2\right) + \beta E\left(\sigma_{t-1}^2\right) + \phi E\left(e_{t-1}\right) \\
\Leftrightarrow \quad & \sigma^2 = K + \alpha E\left[E\left(e_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \beta\sigma^2 + \phi E\left[E\left(e_{t-1} \mid \sigma_{t-1}\right)\right] \\
\Leftrightarrow \quad & \sigma^2 = K + \alpha E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \beta\sigma^2 + \phi E\left[\sigma_{t-1} E\left(Z_{t-1} \mid \sigma_{t-1}\right)\right] \\
\Leftrightarrow \quad & \sigma^2 = K + \alpha E\left[\sigma_{t-1}^2 \cdot 1\right] + \beta\sigma^2 + \phi E\left[\sigma_{t-1} \cdot 0\right] \\
\Leftrightarrow \quad & \sigma^2 = K + \alpha\sigma^2 + \beta\sigma^2 \\
\Leftrightarrow \quad & \sigma^2 = \frac{K}{1 - \alpha - \beta}.
\end{aligned}
$$

5) Consider the GJR-QGARCH model given by

$$e_t = \sigma_t Z_t,$$

$$\sigma_t^2 = K + \delta\sigma_{t-1}^2 + \alpha e_{t-1}^2 + \phi e_{t-1}^2 I_{t-1},$$

and

$$I_{t-1} = \begin{cases} 0, & \text{if } e_{t-1} \geq 0, \\ 1, & \text{if } e_{t-1} < 0. \end{cases}$$

Then, as in problem 1,

$$E\left(e_t\right) = 0$$

and

$$E\left(e_t^2\right) = E\left[\sigma_t^2\right].$$

If stationarity holds

$$\begin{aligned}
& E\left(\sigma_t^2\right) = K + \delta E\left(\sigma_{t-1}^2\right) + \alpha E\left(e_{t-1}^2\right) + \phi E\left(e_{t-1}^2 I_{t-1}\right) \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \mid \sigma_{t-1}\right)\right] + \phi E\left[E\left(e_{t-1}^2 I_{t-1} \mid I_{t-1}\right)\right] \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha E\left[\sigma_{t-1}^2 \cdot 1\right] + \phi E\left(e_{t-1}^2 \Pr\left(e_{t-1} < 0\right)\right) \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha E\left[\sigma_{t-1}^2\right] + \phi E\left[E\left(e_{t-1}^2 \Pr\left(e_{t-1} < 0\right) \mid \sigma_{t-1}\right)\right] \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha\sigma^2 + \phi E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \Pr\left(\sigma_{t-1} Z_{t-1} < 0\right) \mid \sigma_{t-1}\right)\right] \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha\sigma^2 + \phi E\left[\sigma_{t-1}^2 E\left(Z_{t-1}^2 \cdot \frac{1}{2} \mid \sigma_{t-1}\right)\right] \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha\sigma^2 + \phi E\left[\sigma_{t-1}^2 \cdot \frac{1}{2}\right] \\
\Leftrightarrow\quad & \sigma^2 = K + \delta\sigma^2 + \alpha\sigma^2 + \frac{\phi}{2}\sigma^2 \\
\Leftrightarrow\quad & \sigma^2 = \frac{K}{1 - \delta - \alpha - \frac{\phi}{2}}.
\end{aligned}$$