LECTURE CLASS

TUES 6 MARCH

16:00-17:00PM

MATH20802

Announcements

1. Attendance

2. Strike

Ex 1

Suppose $X_1, \ldots, X_n$ IID $Exp(\lambda)$.
Find the MLE of $P(X < 1)$.

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

$$P(X < 1) = \int_0^1 \lambda e^{-\lambda x} \, dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{(-\lambda)} \right]_0^1$$

$$= -\left(e^{-\lambda} - 1\right)$$

$$= 1 - e^{-\lambda}$$

is a 1-to-1 function of λ

$$\Rightarrow \widehat{P(X < 1)} = 1 - e^{-\hat{\lambda}}$$

$$= 1 - e^{-\frac{1}{\bar{X}}}$$

Ex 2

Suppose $X_1, \ldots, X_n$ IID $N(\mu_X, \sigma^2)$

Suppose $Y_1, \ldots, Y_n$ IID $N(\mu_Y, \sigma^2)$.

The samples are indep of each other.

σ^2 is assumed known. Find the MLE of $P(X < Y)$.

$$\widehat{\mu_X} = \overline{X}$$

$$\widehat{\mu_Y} = \overline{Y}$$

$$P(X < Y) = P(X - Y < 0)$$

$$= P(N(\mu_X, \sigma^2) - N(\mu_Y, \sigma^2) < 0)$$

$$= P(N(\mu_X - \mu_Y, 2\sigma^2) < 0)$$

$$= P\left(\boxed{\frac{N(\mu_X - \mu_Y, 2\sigma^2) - (\mu_X - \mu_Y)}{\sqrt{2}\,\sigma}} < \frac{0 - (\mu_X - \mu_Y)}{\sqrt{2}\,\sigma}\right)$$

$$= P\left(N(0,1) < \frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right)$$

$$= \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right)$$

Hence the MLE of $P(X < Y)$ is

$$\Phi\left(\frac{\hat{\mu}_Y - \hat{\mu}_X}{\sqrt{2}\,\sigma}\right)$$

$$= \Phi\left(\frac{\bar{Y} - \bar{X}}{\sqrt{2}\,\sigma}\right).$$

Properties of MLE

1. If $\hat{\theta}$ is an MLE of θ and g is a 1-to-1 function $g(\hat{\theta})$ is an MLE of $g(\theta)$.

[INVARIANCE PRINCIPLE]

2. If $\hat{\theta}$ is an MLE of θ then it is asymptotically unbiased, that is, $\lim_{n \to \infty} \text{Bias}(\hat{\theta}) = 0$

3. If $\hat{\theta}$ is an MLE of θ then it is consistent, that is, $\lim_{n \to \infty} \text{MSE}(\hat{\theta}) = 0$

4. $\sqrt{n}\,(\hat{\theta} - \theta)$ approaches the normal distribution as $n \to \infty$.

Central Limit Theorem

Suppose $X_1, X_2, \ldots, X_n$ is a random sample with $E(X) = \mu$ and $Var(X) = \sigma^2$. Then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0, 1)$$

as $n \longrightarrow \infty$.

Sample mean

Standard Error

CLT does not always hold.

1) $X_1, X_2, \ldots, X_n$ are IID with PDF

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2}, \quad -\infty < x < \infty$$

Cauchy distribution

2) $X_1, X_2, \ldots, X_n$ are IID with PDF

$$f(x) = \frac{1}{x^2} e^{-\frac{1}{x}}, \quad x > 0$$

unit Fréchet distribution.

<u>Home work</u> : Show that CLT does not hold for both these examples.

Sampling Distributions

are distributions that arise as functions of the data.

i) Chi squared distribution (χ^2_a)

ii) Student's t " (t_a)

iii) F " ($F_{a,b}$)

i) Chi square distribution

$X \sim \chi^2_a$ if

$$f_X(x) = \frac{1}{2\Gamma(\frac{a}{2})} \left(\frac{x}{2}\right)^{\frac{a}{2}-1} e^{-\frac{x}{2}}, \quad x > 0, \quad a > 0$$

a = "degree of freedom" parameter

Properties

PROOF 1) $E(\chi^2_a) = a$

" 2) $Var(\chi^2_a) = 2a$

3) if $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$ then

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2_{n-1}$$

PROOF 4). $M(t) = (1-2t)^{-\frac{a}{2}}$

" 5). $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_1$

" 6). If $X_1 \sim \chi^2_a$ and $X_2 \sim \chi^2_b$ are independent then $X_1 + X_2 \sim \chi^2_{a+b}$

PROOF 7). If $Z_1, Z_2, \ldots, Z_n$ are IID $N(0,1)$ then

$$\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$$

" 8) If $X_1, X_2, \ldots, X_n$ are IID $N(\mu, \sigma^2)$ then

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$

Proof of 4

$$M(t) = \int_0^\infty e^{tx} \cdot \frac{1}{2\Gamma(\frac{a}{2})} \left(\frac{x}{2}\right)^{\frac{a}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2^{\frac{a}{2}}\Gamma(\frac{a}{2})} \int_0^\infty x^{\frac{a}{2}-1} e^{-x(\frac{1}{2}-t)} dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= x\left(\tfrac{1}{2}-t\right) \\ x &= \frac{y}{\frac{1}{2}-t} \\ dx &= \frac{dy}{\frac{1}{2}-t} \end{aligned}}$$

$$= \frac{1}{2^{\frac{a}{2}}\Gamma(\frac{a}{2})} \int_0^\infty \left(\frac{y}{\frac{1}{2}-t}\right)^{\frac{a}{2}-1} e^{-y} \frac{dy}{\frac{1}{2}-t}$$

$$= \frac{1}{2^{\frac{a}{2}}\Gamma(\frac{a}{2})\left(\frac{1}{2}-t\right)^{\frac{a}{2}}} \int_0^\infty y^{\frac{a}{2}-1} e^{-y} dy$$

$$= \frac{1}{2^{\frac{a}{2}}\Gamma(\frac{a}{2})\left(\frac{1}{2}-t\right)^{\frac{a}{2}}} \Gamma\left(\frac{a}{2}\right)$$

$$= (1-2t)^{-\frac{a}{2}}.$$