EXAMPLE CLASS

TUES 13 FEB

17:00-18:00PM

MATH20802

Q1

X_1, X_2, X_3 IID $N(\mu, \sigma^2)$

$$\hat{\mu}_1 = \frac{X_1 + 2X_2 + X_3}{4}$$

$$\hat{\mu}_2 = \frac{X_1 + X_2 + X_3}{3}$$

$$\text{Bias}(\hat{\mu}_1) = E(\hat{\mu}_1) - \mu$$

$$= E\left(\frac{X_1 + 2X_2 + X_3}{4}\right) - \mu$$

$$= \frac{1}{4}\left[E(X_1) + 2E(X_2) + E(X_3)\right] - \mu$$

$$= \frac{1}{4}\left[\mu + 2\mu + \mu\right] - \mu$$

$$= 0$$

$$\text{Bias}(\hat{\mu}_2) = E(\hat{\mu}_2) - \mu$$

$$= E\left(\frac{X_1 + X_2 + X_3}{3}\right) - \mu$$

$$= \frac{1}{3}\left[E(X_1) + E(X_2) + E(X_3)\right] - \mu$$

$$= \frac{1}{3}\left[\mu + \mu + \mu\right] - \mu$$

$$= 0$$

There is nothing to choose with respect to bias.

$$MSE(\hat{\mu}_1) = Var(\hat{\mu}_1)$$

$$= Var\left[\frac{X_1 + 2X_2 + X_3}{4}\right]$$

$$= \frac{1}{16}\left[Var(X_1) + 4\,Var(X_2) + Var(X_3)\right]$$

$$= \frac{1}{16}\left[\sigma^2 + 4\sigma^2 + \sigma^2\right]$$

$$= \frac{3\sigma^2}{8}$$

$$MSE(\hat{\mu}_2) = Var(\hat{\mu}_2)$$

$$= Var\left[\frac{X_1 + X_2 + X_3}{3}\right]$$

$$= \frac{1}{9}\left[Var(X_1) + Var(X_2) + Var(X_3)\right]$$

$$= \frac{1}{9}\left[\sigma^2 + \sigma^2 + \sigma^2\right]$$

$$= \frac{\sigma^2}{3}$$

Clearly, $\frac{\sigma^2}{3} < \frac{3\sigma^2}{8}$.

So, we prefer $\hat{\mu}_2$ to $\hat{\mu}_1$.

Q2 $X_1, X_2, \ldots, X_n$ IID with $E(X_i) = \mu$
$Var(X_i) = \sigma^2$

(i) $Bias\left(\sum_{i=1}^{n} a_i X_i\right) = 0$

$$\Leftrightarrow E\left(\sum_{i=1}^{n} a_i X_i\right) = \mu$$

$$\Leftrightarrow \sum_{i=1}^{n} a_i E(X_i) = \mu$$

$$\Leftrightarrow \sum_{i=1}^{n} a_i \mu = \mu$$

$$\Leftrightarrow \cancel{\mu} \sum_{i=1}^{n} a_i = \cancel{\mu}$$

$$\Leftrightarrow \sum_{i=1}^{n} a_i = 1$$

(ii) $\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right)$

$$= \sum_{i=1}^{n} a_i^2 \, \mathrm{Var}(X_i)$$

$$= \sum_{i=1}^{n} a_i^2 \, \sigma^2$$

$$= \sigma^2 \sum_{i=1}^{n} a_i^2$$

Hint $\sigma^2 \left[\left(\sum_{i=1}^{n} \boxed{\left(a_i - \frac{1}{n}\right)^2} \right) + \frac{1}{n} \right]$

≥ 0

will be minimized if

$$\left(a_i - \frac{1}{n}\right)^2 = 0 \quad \forall i$$

$$\Leftrightarrow \quad a_i - \frac{1}{n} = 0 \quad \forall i$$

$$\Leftrightarrow \quad a_i = \frac{1}{n} \quad \forall i$$

The estimator giving minimum variance is

$$\sum_{i=1}^{n} \frac{1}{n} X_i = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}.$$

Q 3

$$\boxed{\begin{aligned} E(\chi^2_\nu) &= \nu \\ Var(\chi^2_\nu) &= 2\nu \end{aligned}}$$

Q3

$$S'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

"sample variance"

$$E(S'^2) = E\left[\frac{(n-1)S'^2}{\sigma^2}\cdot\frac{\sigma^2}{n-1}\right]$$

$$= \frac{\sigma^2}{n-1}E\left[\frac{(n-1)S'^2}{\sigma^2}\right]$$

Hint

$$\frac{\sigma^2}{n-1}E\left[\chi^2_{n-1}\right]$$

$$= \frac{\sigma^2}{n-1}\cdot n-1$$

$$= \sigma^2$$

$$\Rightarrow \text{Bias}(S'^2) = 0$$

$$Var(S'^2) = Var\left[\frac{(n-1)S'^2}{\sigma^2}\cdot\frac{\sigma^2}{n-1}\right]$$

$$= \frac{\sigma^4}{(n-1)^2}\cdot Var\left[\frac{(n-1)S'^2}{\sigma^2}\right]$$

Hint

$$\frac{\sigma^4}{(n-1)^2}\cdot Var\left[\chi^2_{n-1}\right]$$

$$= \frac{\sigma^4}{(n-1)^2}\cdot 2(n-1)$$

$$= \frac{2\sigma^4}{n-1}$$

$$\Rightarrow MSE(S'^2) = \frac{2\sigma^4}{n-1} \longrightarrow 0 \text{ as } n\to\infty$$

$\Rightarrow S'^2$ is consistent.

Q4 (i)

$$\hat{\gamma} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$$

$$\text{Bias}(\hat{\gamma}) = E(\hat{\gamma}) - \gamma$$

$$= \left(\frac{1}{2n}\sum_{i=1}^{n} E(X_i^2)\right) - \gamma$$

$$E(X_i^2) = \int_0^\infty x^2 \cdot \frac{x}{\gamma} e^{-\frac{x^2}{2\gamma}}\, dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= \frac{x^2}{2\gamma} \\ x &= \sqrt{2\gamma y} \\ \frac{dx}{dy} &= \sqrt{\frac{\gamma}{2y}} \end{aligned}}$$

$$= \int_0^\infty 2\gamma y \cdot \frac{\sqrt{2\gamma y}}{\gamma} e^{-y} \sqrt{\frac{\gamma}{2y}}\, dy$$

$$= 2\gamma \int_0^\infty y e^{-y}\, dy$$

$$= 2\gamma\, \Gamma(2)$$

$$= 2\gamma$$

$$\Rightarrow \text{Bias}(\hat{\gamma}) = \left(\frac{1}{2n} \cdot \sum_{i=1}^{n} 2\gamma\right) - \gamma$$

$$= \frac{1}{2n} \cdot 2\gamma \cdot n - \gamma$$

$$= 0$$

$\Rightarrow$ $\hat{\gamma}$ is unbiased.