EXAMPLE CLASS

THURS 1 FEB

16:00-17:00PM

MATH20802

Q1 $X \sim N(\mu, \sigma^2)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 tx - x^2 + 2\mu x - \mu^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2 t)x + \mu^2}{2\sigma^2}} dx$$

Completion of squares

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2 - (\mu+\sigma^2 t)^2 + \mu^2}{2\sigma^2}} dx$$

$$= e^{\frac{(\mu+\sigma^2 t)^2 - \mu^2}{2\sigma^2}} \cdot \underbrace{\left[\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}} dx \right]}_{1}$$

because it is an integral of the PDF of $N(\mu+\sigma^2 t, \sigma^2)$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Q2 Show that

$$aX_1 + bX_2 + c \sim N(a\mu_1 + b\mu_2 + c, \; a^2\sigma_1^2 + b^2\sigma_2^2)$$

MGF of the LHS

$$= E\left[e^{t(aX_1 + bX_2 + c)}\right]$$

$$\overset{\text{indep}}{=} E\left[e^{taX_1}\right] E\left[e^{tbX_2}\right] e^{tc}$$

$$= M_{X_1}(ta) \; M_{X_2}(tb) \; e^{tc}$$

$$\overset{Q1}{=} e^{\mu_1 ta + \frac{\sigma_1^2 (ta)^2}{2}} e^{\mu_2 tb + \frac{\sigma_2^2 (tb)^2}{2}} e^{tc}$$

$$= e^{(\mu_1 a + \mu_2 b + c)t + \frac{(\sigma_1^2 a^2 + \sigma_2^2 b^2)t^2}{2}}$$

$$= \text{MGF of } N(\mu_1 a + \mu_2 b + c, \; \sigma_1^2 a^2 + \sigma_2^2 b^2)$$

$$= \text{MGF of the RHS.}$$

Q3 Show that

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

MGF of the LHS

$$= E\left[e^{t\bar{X}}\right]$$

$$= E\left[e^{t\frac{X_1 + \cdots + X_n}{n}}\right]$$

$$\overset{\text{indep}}{=} E\left[e^{\frac{tX_1}{n}}\right] \cdots E\left[e^{\frac{tX_n}{n}}\right]$$

$$= M_{X_1}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right)$$

$$\overset{Q1}{=} e^{\mu\frac{t}{n} + \frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}} \cdots e^{\mu\frac{t}{n} + \frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2n^2} \cdot n}$$

$$= e^{\mu t + \frac{\left(\frac{\sigma^2}{n}\right)t^2}{2}}$$

$\overset{Q1}{=}$ MGF of $N\left(\mu, \frac{\sigma^2}{n}\right)$

$=$ MGF of the RHS.

Q4 $X \sim Uni[a, b]$

$$M_X(t) = \int_a^b e^{tx} \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx} \cdot dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b = \frac{1}{b-a} \cdot \frac{e^{tb} - e^{ta}}{t}$$

Q5 $X \sim Exp(\lambda)$

$$M_X(t) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} \, dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty$$

$$= \lambda \left[0 - \frac{1}{-(\lambda - t)} \right] \quad \text{if} \quad \lambda > t$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if} \quad \lambda > t$$

Q6 $X \sim Ga(a, \lambda)$

$$M_X(t) = \int_0^\infty e^{tx} \cdot \frac{\lambda^a x^{a-1} e^{-\lambda x}}{\Gamma(a)} \, dx$$

$$= \lambda^a \int_0^\infty \frac{x^{a-1} e^{-(\lambda - t)x}}{\Gamma(a)} \, dx$$

$$= \frac{\lambda^a}{(\lambda - t)^a} \int_0^\infty \frac{(\lambda - t)^a x^{a-1} e^{-(\lambda - t)x}}{\Gamma(a)} \, dx$$

$\int_0^\infty \frac{(\lambda - t)^a x^{a-1} e^{-(\lambda - t)x}}{\Gamma(a)} \, dx = 1$ (PDF of $Ga(a, \lambda - t)$)

$$= \left(\frac{\lambda}{\lambda - t}\right)^a .$$