MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 7

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$ is a random sample from a distribution specified by the joint probability mass function

$$p(x, y) = \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y},$$

where p and q satisfying $0 < p < 1$, $0 < q < 1$ and $0 < p + q < 1$ are unknown parameters. The joiny likelihood function of p and q is

$$\begin{aligned} L(p, q) &= \prod_{i=1}^m \left[\frac{n!}{X_i! Y_i! \left(n - X_i - Y_i\right)!} p^{X_i} q^{Y_i} (1-p-q)^{n-X_i-Y_i} \right] \\ &= \frac{(n!)^m}{\prod_{i=1}^m X_i! \prod_{i=1}^m Y_i! \prod_{i=1}^m \left(n - X_i - Y_i\right)!} p^{\sum_{i=1}^m X_i} q^{\sum_{i=1}^m Y_i} (1-p-q)^{mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i}. \end{aligned}$$

The log likelihood function is

$$\begin{aligned} \log L(p, q) &= m \log n! - \sum_{i=1}^m \log X_i! - \sum_{i=1}^m \log Y_i! - \sum_{i=1}^m \log \left(n - X_i - Y_i\right)! \\ &\quad + \left(\sum_{i=1}^m X_i\right) \log p + \left(\sum_{i=1}^m Y_i\right) \log q + \left(mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i\right) \log(1-p-q). \end{aligned}$$

The partial derivatives with respect to p and q are

$$\frac{\partial \log L(p, q)}{\partial p} = \left(\sum_{i=1}^m X_i\right) \frac{1}{p} - \left(mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i\right) \frac{1}{1-p-q} \tag{1}$$

and

$$\frac{\partial \log L(p, q)}{\partial q} = \left(\sum_{i=1}^m Y_i\right) \frac{1}{q} - \left(mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i\right) \frac{1}{1-p-q}. \tag{2}$$

Setting to zero, equation (1) can be rewritten as

$$(1-p-q) \left(\sum_{i=1}^m X_i\right) = p \left(mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i\right). \tag{3}$$

Setting to zero, equation (2) can be rewritten as

$$(1-p-q) \left(\sum_{i=1}^m Y_i\right) = q \left(mn - \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i\right). \tag{4}$$

Summing equations (3) and (4) gives

$$(1-p-q)\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} Y_i\right) = (p+q)\left(mn - \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} Y_i\right),$$

which gives

$$p+q = \frac{1}{mn}\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} Y_i\right). \tag{5}$$

Substituting (5) into (3) gives the estimator of p as

$$\widehat{p} = \frac{1}{mn}\sum_{i=1}^{m} X_i.$$

Substituting (5) into (4) gives the estimator of q as

$$\widehat{q} = \frac{1}{mn}\sum_{i=1}^{m} Y_i.$$