MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 4

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $\text{Uni}[0, \theta]$. Let $Z = \max(X_1, X_2, \ldots, X_n)$. The cdf of Z is

$$\begin{aligned} F_Z(z) &= \Pr\left[\max(X_1, X_2, \ldots, X_n) \leq z\right] \\ &= \Pr\left[X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z\right] \\ &= \Pr\left[X_1 \leq z\right] \Pr\left[X_2 \leq z\right] \cdots \Pr\left[X_n \leq z\right] \\ &= \frac{z}{\theta} \frac{z}{\theta} \cdots \frac{z}{\theta} \\ &= \frac{z^n}{\theta^n}. \end{aligned}$$

So the pdf of Z is

$$f_Z(z) = \frac{nz^{n-1}}{\theta^n}$$

for $0 < z < \theta$. Hence, the bias of Z as an estimator of θ is

$$\begin{aligned} \text{Bias}(Z) &= E(Z) - \theta \\ &= \int_0^\theta z \frac{nz^{n-1}}{\theta^n} dz - \theta \\ &= \frac{n}{\theta^n} \int_0^\theta z^n dz - \theta \\ &= \frac{n}{\theta^n} \left[\frac{z^{n+1}}{n+1}\right]_0^\theta - \theta \\ &= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} - \theta \\ &= \frac{n\theta}{n+1} - \theta \\ &= -\frac{\theta}{n+1}. \end{aligned}$$

Hence, Z is a biased estimator for θ.