MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 3

Suppose X is a random variable with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right]$$

for $x > 0$, $-\infty < \mu < +\infty$ and $\sigma > 0$. Then

$$\begin{aligned}
E\left(X^n\right) &= \int_0^\infty x^n \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty x^{n-1} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[(n-1)y\right] \exp\left[-\frac{(y - \mu)^2}{2\sigma^2}\right] \exp(y) dy \qquad [\text{Set } y = \log x] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[ny - \frac{(y - \mu)^2}{2\sigma^2}\right] dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[-\frac{(y - \mu)^2 - 2\sigma^2 ny}{2\sigma^2}\right] dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[-\frac{y^2 + \mu^2 - 2\mu y - 2\sigma^2 ny}{2\sigma^2}\right] dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[-\frac{y^2 + \mu^2 - 2\left(\mu + \sigma^2 n\right) y}{2\sigma^2}\right] dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[-\frac{\left[y - \left(\mu + \sigma^2 n\right)\right]^2 + \mu^2 - \left(\mu + \sigma^2 n\right)^2}{2\sigma^2}\right] dy \\
&= \exp\left[-\frac{\mu^2 - \left(\mu + \sigma^2 n\right)^2}{2\sigma^2}\right] \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \exp\left[-\frac{\left[y - \left(\mu + \sigma^2 n\right)\right]^2}{2\sigma^2}\right] dy \\
&= \exp\left[-\frac{\mu^2 - \left(\mu + \sigma^2 n\right)^2}{2\sigma^2}\right] \\
&= \exp\left[\frac{2\mu\sigma^2 n + \sigma^4 n^2}{2\sigma^2}\right] \\
&= \exp\left[\mu n + \frac{\sigma^2 n^2}{2}\right].
\end{aligned}$$