MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 9

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, \sigma^2)$, where σ^2 is assumed known.

1. The power function, $\Pi(\theta)$, for $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$ is

$$
\begin{aligned}
\Pi(\theta) &= \Pr\left(\frac{\sqrt{n}}{\sigma}\left(\bar{x} - \theta_0\right) \leq -z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} - \theta_0 \leq -\frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} \leq \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} - \theta \leq \theta_0 - \theta - \frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\frac{\sqrt{n}}{\sigma}\left(\bar{x} - \theta\right) \leq \frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) - z_\alpha \middle| \theta \right) \\
&= \Pr\left(Z \leq \frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) - z_\alpha \right) \\
&= \Phi\left(\frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) - z_\alpha \right).
\end{aligned}
$$

2. The power function, $\Pi(\theta)$, for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ is

$$
\begin{aligned}
\Pi(\theta) &= \Pr\left(\frac{\sqrt{n}}{\sigma}\left(\bar{x} - \theta_0\right) \geq z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} - \theta_0 \geq \frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} \geq \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\bar{x} - \theta \geq \theta_0 - \theta + \frac{\sigma}{\sqrt{n}} z_\alpha \middle| \theta \right) \\
&= \Pr\left(\frac{\sqrt{n}}{\sigma}\left(\bar{x} - \theta\right) \geq \frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) + z_\alpha \middle| \theta \right) \\
&= \Pr\left(Z \geq \frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) + z_\alpha \right) \\
&= 1 - \Pr\left(Z < \frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) + z_\alpha \right) \\
&= 1 - \Phi\left(\frac{\sqrt{n}}{\sigma}\left(\theta_0 - \theta\right) + z_\alpha \right).
\end{aligned}
$$

Note that we have used the fact $(\sqrt{n}/\sigma)(\bar{x}-\theta) = Z \sim N(0,1)$. Furthermore, $\Phi(\cdot)$ denotes the cumulative distribution function of $N(0,1)$.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with parameter p. Assuming a significance level of α and that $\overline{X} = (X_1+X_2+\cdots+X_n)/n$ has an approximate normal distribution,

3. The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p < p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}(\bar{x}-p_0) < -z_\alpha \middle| p\right) \\
&= \Pr\left(\bar{x} < p_0 - \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_\alpha \middle| p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \middle| p\right) \\
&= \Pr\left(Z < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \middle| p\right) \\
&= \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

4. The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p > p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}(\bar{x}-p_0) > z_\alpha \middle| p\right) \\
&= \Pr\left(\bar{x} > p_0 + \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_\alpha \middle| p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \middle| p\right) \\
&= \Pr\left(Z > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \middle| p\right) \\
&= 1 - \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.