MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 8

1. Let X_1, X_2, X_3 be a random sample from the $Po(\lambda)$ distribution.

   (i) Note that

$$
\begin{aligned}
L(1) &= \prod_{i=1}^{3} \frac{1^{X_i} \exp(-1)}{X_i!} \\
&= \frac{1^{\sum_{i=1}^{3} X_i} \exp(-1)}{\prod_{i=1}^{3} X_i!} \\
&= \frac{\exp(-3)}{\prod_{i=1}^{3} X_i!}
\end{aligned}
$$

   and

$$
\begin{aligned}
L(2) &= \prod_{i=1}^{3} \frac{2^{X_i} \exp(-2)}{X_i!} \\
&= \frac{2^{\sum_{i=1}^{3} X_i} \exp(-2)}{\prod_{i=1}^{3} X_i!} \\
&= \frac{2^{\sum_{i=1}^{3} X_i} \exp(-6)}{\prod_{i=1}^{3} X_i!}.
\end{aligned}
$$

   By the N–P lemma, the most powerful test is to reject H_0 if and only if $L(1)/L(2) = \exp(3)/2^{\sum_{i=1}^{3} X_i} < c$ for some c.

   (ii) One can rearrange $\exp(3)/2^{\sum_{i=1}^{3} X_i} < c$ as $T = X_1 + X_2 + X_3 \geq k$ for some k.

   (iii) If $\alpha = 0.1$ then

$$
\begin{aligned}
\Pr(\text{Type I Error}) &= \Pr(T > k \mid \lambda = 1) \\
&= 1 - \Pr(Po(3) \leq k) \\
&= 1 - \sum_{i=0}^{k} \frac{3^i \exp(-3)}{i!} \\
&= 0.1.
\end{aligned}
$$

   If $k = 4$ we get $\Pr(T > 4) = 0.185$. If $k = 5$ we get $\Pr(T > 5) = 0.084$. So, our test is to reject H_0 if and only if $T > 5$ with the significance level of 0.084.

2. (i) Note that

$$
\begin{aligned}
L(\sigma_0) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{X_i^2}{2\sigma_0^2}\right) \\
&= \frac{1}{(\sqrt{2\pi}\sigma_0)^n} \exp\left(-\frac{\sum_{i=1}^{n} X_i^2}{2\sigma_0^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
L(\sigma_1) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{X_i^2}{2\sigma_1^2}\right) \\
&= \frac{1}{(\sqrt{2\pi}\sigma_1)^n} \exp\left(-\frac{\sum_{i=1}^{n} X_i^2}{2\sigma_1^2}\right).
\end{aligned}
$$

By the N–P lemma, the most powerful test is to reject H_0 if and only if $L(\sigma_0)/L(\sigma_1) = (\sigma_1/\sigma_0)^n \exp\{(1/2)(1/\sigma_1^2 - 1/\sigma_0^2)\sum_{i=1}^{n} X_i^2\} < c$. Since $\sigma_1 > \sigma_0$ this condition is equivalent to $T = \sum_{i=1}^{n}(X_i/\sigma_0)^2 > k$ for some k.

(ii) We need $\Pr(\text{Reject } H_0 \mid \sigma = \sigma_0) = 0.05$. When $\sigma = \sigma_0$, $T \sim \chi_n^2$, so $k = \chi_{n,0.05}^2$.

(iii) If $n = 10$, $\sigma_0 = 2$ and $\sigma_1 = 3$ then

$$
\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(\text{Accept } H_0 \mid \sigma = 3) \\
&= \Pr(T \leq \chi_{10,0.05}^2 \mid \sigma = 3) \\
&= \Pr((2/3)^2 T \leq (2/3)^2 \chi_{10,0.05}^2 \mid \sigma = 3) \\
&= \Pr(\chi_{10}^2 \leq (2/3)^2 \chi_{10,0.05}^2) \\
&= \Pr(\chi_{10}^2 \leq 8.136461).
\end{aligned}
$$