MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 7

1. Since $(X_i - \mu) = \left(X_i - \overline{X}\right) + \left(\overline{X} - \mu\right)$, we can write

$$\begin{aligned}
\sum_{i=1}^{n} (X_i - \mu)^2 &= \sum_{i=1}^{n} \left[\left(X_i - \overline{X}\right) + \left(\overline{X} - \mu\right)\right]^2 \\
&= \sum_{i=1}^{n} \left[\left(X_i - \overline{X}\right)^2 + 2\left(X_i - \overline{X}\right)\left(\overline{X} - \mu\right) + \left(\overline{X} - \mu\right)^2\right] \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + \sum_{i=1}^{n} 2\left(\overline{X} - \mu\right)\left(X_i - \overline{X}\right) + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + 2\left(\overline{X} - \mu\right)\sum_{i=1}^{n} \left(X_i - \overline{X}\right) + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + 2\left(\overline{X} - \mu\right)\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X}\right) + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + 2\left(\overline{X} - \mu\right)\left(\sum_{i=1}^{n} X_i - n\overline{X}\right) + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + 2\left(\overline{X} - \mu\right)\left(\sum_{i=1}^{n} X_i - n\frac{1}{n}\sum_{i=1}^{n} X_i\right) + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + \sum_{i=1}^{n} \left(\overline{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 + n\left(\overline{X} - \mu\right)^2.
\end{aligned}$$

2. Let $Y = Z^2$. Then

$$\begin{aligned}
\Pr(Y < y) &= \Pr\left(N^2(0,1) < y\right) \\
&= \Pr\left(-\sqrt{y} < N(0,1) < \sqrt{y}\right) \\
&= \Pr\left(N(0,1) < \sqrt{y}\right) - \Pr\left(N(0,1) < -\sqrt{y}\right) \\
&= \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right).
\end{aligned}$$

Differentiating with respect to y,

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}}\phi\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}}\phi\left(-\sqrt{y}\right) \\
&= \frac{1}{\sqrt{y}}\phi\left(\sqrt{y}\right) \\
&= \frac{1}{\sqrt{2\pi y}}\exp\left(-\frac{y}{2}\right),
\end{aligned}$$

the pdf of a χ_1^2 random variable.

3. Note that

$$\begin{aligned}\Pr(T < t) &= \int_0^\infty \Pr\left(N(0,1) < t\sqrt{\frac{x}{\nu}}\right) f_{\chi^2_\nu}(x)dx \\ &= \int_0^\infty \Phi\left(t\sqrt{\frac{x}{\nu}}\right) f_{\chi^2_\nu}(x)dx.\end{aligned}$$

Differentiating with respect to t,

$$\begin{aligned}f_Y(y) &= \int_0^\infty \phi\left(t\sqrt{\frac{x}{\nu}}\right)\sqrt{\frac{x}{\nu}} f_{\chi^2_\nu}(x)dx \\ &= \frac{1}{\sqrt{2\pi\nu}2^{\nu/2}\Gamma(\nu/2)}\int_0^\infty x^{(\nu-1)/2}\exp\left\{-\frac{1}{2}\left(1+\frac{t^2}{\nu}\right)x\right\}dx \\ &= \frac{1}{\sqrt{2\pi\nu}2^{\nu/2}\Gamma(\nu/2)}2^{(\nu+1)/2}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}\int_0^\infty w^{(\nu-1)/2}\exp\left(-w\right)dw \\ &= \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi\nu}\Gamma(\nu/2)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2},\end{aligned}$$

the pdf of a t_ν random variable.

4. Using the fact $19S^2/1.4 \sim \chi^2_{19}$, we can write

$$\begin{aligned}&\Pr(\chi^2_{19,0.975} < \frac{19S^2}{1.4} < \chi^2_{19,0.025}) = 0.95 \\ \Leftrightarrow\ &\Pr(8.907 < \frac{19S^2}{1.4} < 32.852) = 0.95 \\ \Leftrightarrow\ &\Pr(\frac{1.4\times 8.907}{19} < S^2 < \frac{1.4\times 32.852}{19}) = 0.95 \\ \Leftrightarrow\ &\Pr(0.656 < S^2 < 2.421) = 0.95,\end{aligned}$$

so $a = 0.656$ and $b = 2.421$.

5. Assume the 16 observations are a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ^2 are unknown. We want to test $H_0 : \sigma^2 = 4.5^2 = \sigma_0^2$, say, versus $H_1 : \sigma^2 \neq 4.5^2$. The standard test is to reject H_0 if $(n-1)S^2/\sigma_0^2 < \chi_{15,0.975} = 6.262$ or $(n-1)S^2/\sigma_0^2 > \chi_{15,0.025} = 27.488$. For the given data, $(n-1)S^2/\sigma_0^2 = 15 \times 4.84/20.25 = 3.585$. Therefore, we reject H_0 at the 5 percent level and conclude that $\sigma^2 \neq 4.5^2$.

6. If $X \sim F_{\nu_1,\nu_2}$ then we can write $X = (X_1/\nu_1)/(X_2/\nu_2)$, where $X_1 \sim \chi^2_{\nu_1}$ and $X_2 \sim \chi^2_{\nu_2}$. So, $1/X = (X_2/\nu_2)/(X_1/\nu_1) \sim F_{\nu_2,\nu_1}$.

   I defined $F_{\nu_1,\nu_2,\alpha}$ as

$$\Pr\left(F_{\nu_1,\nu_2} < F_{\nu_1,\nu_2,\alpha}\right) = 1-\alpha, \tag{1}$$

   this definition may be different from those you saw in other courses. It follows that

$$\Pr\left(\frac{1}{F_{\nu_1,\nu_2}} > \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = 1-\alpha$$

$$\implies \quad 1 - \Pr\left(\frac{1}{F_{\nu_1,\nu_2}} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = 1 - \alpha$$

$$\implies \quad \Pr\left(\frac{1}{F_{\nu_1,\nu_2}} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = \alpha$$

$$\implies \quad \Pr\left(F_{\nu_2,\nu_1} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = \alpha \qquad \text{since } F_{\nu_2,\nu_1} = \tfrac{1}{F_{\nu_1,\nu_2}}$$

$$\implies \quad F_{\nu_2,\nu_1,1-\alpha} = \frac{1}{F_{\nu_1,\nu_2,\alpha}} \qquad \text{by definition of } F_{\nu_2,\nu_1,1-\alpha}$$