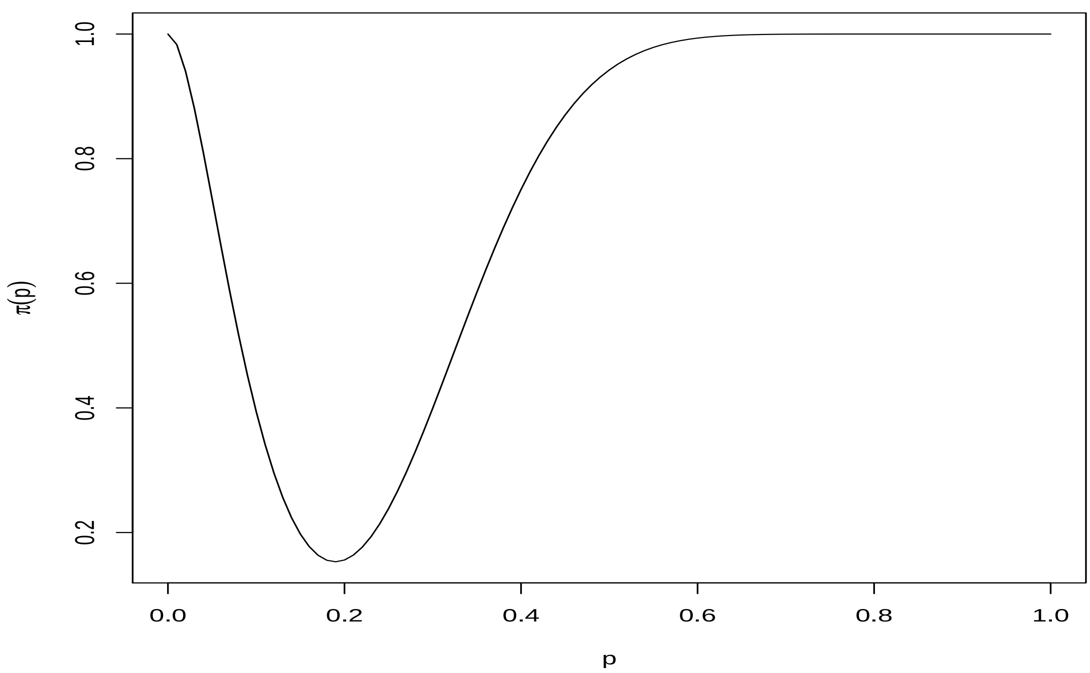
MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 5

1. The power function is

$$\begin{aligned}
\pi(p) &= \Pr(\text{Reject } H_0 \mid p) \\
&= \Pr Y \geq 7 \text{ or } Y \leq 1 \mid p) \\
&= 1 - \Pr(2 \leq Y \leq 6 \mid p) \\
&= 1 - \left[\binom{20}{2} p^2 (1-p)^{18} + \ldots + \binom{20}{6} p^6 (1-p)^{14} \right].
\end{aligned}$$

For $p = 0$, $\pi(0) = 1 - 0 = 1$. For $p = 0.2$, $\pi(0.2) = 0.1558$. For $p = 0.4$, $\pi(0.4) = 0.7505$. For $p = 0.6$, $\pi(0.6) = 0.9935$. For $p = 0.8$, $\pi(0.8) = 1$. For $p = 1$, $\pi(1) = 1$. The level of significance of the test is $\pi(0.2) = 0.1558$. The figure below shows the graph of $\pi(p)$ versus p.



2. Clearly $X \sim Geom(p)$. So,

$$\begin{aligned}
\Pr(\text{Type I Error}) &= \Pr(\text{Reject } H_0 \mid p = 0.1) \\
&= \Pr(X \leq 5 \mid p = 0.1) \\
&= \sum_{k=1}^{5} (0.9)^{k-1} 0.1 \\
&= 0.4095
\end{aligned}$$

and

$$\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(\text{Accept } H_0 \mid p = 0.2) \\
&= \Pr(X > 5 \mid p = 0.2) \\
&= 1 - \sum_{k=1}^{5} (0.8)^{k-1} 0.2 \\
&= 0.3277.
\end{aligned}$$

3. The standard test procedure is to reject H_0 if and only if $\sqrt{n}(\bar{X} - 0)/2 > z_\alpha$ or equivalently $\bar{X} > 2z_\alpha/\sqrt{n}$. So,

$$\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(\text{Accept } H_0 \mid H_1 \text{ is true}) \\
&= \Pr(\bar{X} < 2z_\alpha/\sqrt{n} \mid \mu = 4) \\
&= \Pr(\sqrt{n}(\bar{X} - 4)/2 < z_\alpha - 2\sqrt{n} \mid \mu = 4) \\
&= \Pr(Z < z_\alpha - 2\sqrt{n}) \\
&= \Phi(z_\alpha - 2\sqrt{n}),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

4. The standard test procedure is to reject H_0 if and only if $\mid \sqrt{n}(\bar{X} - 0)/2 \mid > z_{\alpha/2}$ or equivalently $\bar{X} > 2z_{\alpha/2}/\sqrt{n}$ or $\bar{X} < -2z_{\alpha/2}/\sqrt{n}$. So,

$$\begin{aligned}
\Pi(\mu) &= \Pr(\text{Reject } H_0 \mid \mu) \\
&= \Pr(\bar{X} > 2z_{\alpha/2}/\sqrt{n} \text{ or } \bar{X} < -2z_{\alpha/2}/\sqrt{n} \mid \mu) \\
&= \Pr(\sqrt{n}(\bar{X} - \mu)/2 > z_{\alpha/2} - \mu\sqrt{n}/2 \text{ or } \sqrt{n}(\bar{X} - \mu)/2 < -z_{\alpha/2} - \mu\sqrt{n}/2 \mid \mu) \\
&= \Pr(Z < -z_{\alpha/2} - \mu\sqrt{n}/2) + \Pr(Z > z_{\alpha/2} - \mu\sqrt{n}/2) \\
&= \Phi(-z_{\alpha/2} - \mu\sqrt{n}/2) + 1 - \Phi(z_{\alpha/2} - \mu\sqrt{n}/2),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

5. The standard test procedure is to reject H_0 if and only if $\sqrt{n}(\bar{X} - 100)/\sqrt{20} < -z_\alpha$. We need

$$\begin{aligned}
0.9 &\le \Pr(\text{Reject } H_0 \mid \mu = 95) \\
&= \Pr(\bar{X} < 100 - 1.645\sqrt{20}/\sqrt{n} \mid \mu = 95) \\
&= \Pr(\sqrt{n}(\bar{X} - 95)/\sqrt{20} < 5\sqrt{n}/\sqrt{20} - 1.645 \mid \mu = 95) \\
&= \Pr(Z < 5\sqrt{n}/\sqrt{20} - 1.645) \\
&= \Phi(5\sqrt{n}/\sqrt{20} - 1.645).
\end{aligned}$$

From the tables, $\Phi(1.28) = 0.9$, so $5\sqrt{n}/\sqrt{20} - 1.645 = 1.28$, implying $n = 6.85 \approx 7$.

6. The standard test procedure is to reject H_0 if and only if $(n-1)S^2/\sigma_0^2 > \chi^2_{n-1,\alpha}$, where S^2 is the sample variance. With the given values for n, σ_0 and α, it amounts to rejecting H_0 if and only if $S^2 > 1.429 \times 10^{-4}$. So,

$$\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(\text{Accept } H_0 \mid H_1 \text{ is true}) \\
&= \Pr(S^2 < 1.429 \times 10^{-4} \mid \sigma = 2 \times 10^{-2}) \\
&= \Pr(34S^2/(4 \times 10^{-4}) < 12.15 \mid \sigma = 2 \times 10^{-2}) \\
&= \Pr(\chi^2_{34} < 12.15).
\end{aligned}$$