MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 4

1. If $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the pdf $f(x) = \theta x^{\theta-1}$ then the likelihood function for θ is:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \theta X_i^{\theta-1} \\ &= \theta^n \left(\prod_{i=1}^{n} X_i \right)^{\theta-1} \end{aligned}$$

and so the log–likelihood function is:

$$l(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log X_i.$$

The first derivative of $l(\theta)$ is

$$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i$$

and setting this to zero gives the solution $\hat{\theta} = -n / \sum_{i=1}^{n} \log X_i$. This is indeed the mle since the second derivative

$$\frac{d^2 l(\theta)}{d\theta^2} = -\frac{n}{\theta^2} < 0.$$

2. If $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the pdf $f(x) = \theta^2 x \exp(-\theta x)$ then the likelihood function for θ is:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \theta^2 X_i \exp(-\theta X_i) \\ &= \theta^{2n} \left(\prod_{i=1}^{n} X_i \right) \exp\left(-\theta \sum_{i=1}^{n} X_i \right) \end{aligned}$$

and so the log–likelihood function is:

$$l(\theta) = 2n \log \theta + \log \left(\prod_{i=1}^{n} X_i \right) - \theta \sum_{i=1}^{n} X_i.$$

The first derivative of $l(\theta)$ is

$$\frac{dl(\theta)}{d\theta} = \frac{2n}{\theta} - \sum_{i=1}^{n} X_i$$

and setting this to zero gives the solution $\hat{\theta} = 2n / \sum_{i=1}^{n} X_i = 2/\bar{X}$. This is indeed the mle since the second derivative

$$\frac{d^2 l(\theta)}{d\theta^2} = -\frac{2n}{\theta^2} < 0.$$

3. We know that the mle of μ is $\hat{\mu} = \bar{X}$. For the given data, $\bar{X} = 0.903$. Since $\Pr(X < 0) = \Phi(-\mu) = 1 - \Phi(\mu)$, by the invariance principle, the mle of $\Pr(X < 0)$ is $1 - \Phi(\hat{\mu}) = 1 - \Phi(0.903) = 0.1833$. Note $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

4. If $X_1, X_2, \ldots, X_n$ are iid $N(\mu_1, 1)$ then we know that the mle $\hat{\mu}_1 = \bar{X}$. Similarly, if $Y_1, Y_2, \ldots, Y_n$ are iid $N(\mu_2, 1)$ then we know that the mle $\hat{\mu}_2 = \bar{Y}$. So, it follows that $\hat{\alpha} + \hat{\beta} = \bar{X}$ and $\hat{\alpha} - \hat{\beta} = \bar{Y}$. Solving these two equations, we have $\hat{\alpha} = (\bar{X} + \bar{Y})/2$ and $\hat{\beta} = (\bar{X} - \bar{Y})/2$.

5. If $X_1, X_2, \ldots, X_n$ be a random sample from the $Ga(r, \lambda)$ (where r is known) then the likelihood function of λ is

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} \left\{ \Gamma^{-1}(r) \lambda^r X_i^{r-1} \exp(-\lambda X_i) \right\} \\ &= \Gamma^{-n}(r) \lambda^{rn} \left(\prod_{i=1}^{n} X_i \right)^{r-1} \exp\left(-\lambda \sum_{i=1}^{n} X_i \right) \end{aligned}$$

and so the log–likelihood function is:

$$l(\lambda) = -n \log \Gamma(r) + rn \log \lambda + (r-1) \sum_{i=1}^{n} \log X_i - \lambda \sum_{i=1}^{n} X_i.$$

The first derivative of $l(\lambda)$ is

$$\frac{dl(\lambda)}{d\lambda} = \frac{rn}{\lambda} - \sum_{i=1}^{n} X_i$$

and setting this to zero gives the solution $\hat{\lambda} = rn / \sum_{i=1}^{n} X_i = r/\bar{X}$. This is indeed the mle since the second derivative

$$\frac{d^2 l(\lambda)}{d\lambda^2} = -\frac{rn}{\lambda^2} < 0.$$

6. If $X_1, X_2, \ldots, X_n$ are iid $Exp(\lambda_1)$ and $Y_1, Y_2, \ldots, Y_m$ are iid $Exp(\lambda_1 \lambda_2)$ then the likelihood function of (λ_1, λ_2) is

$$\begin{aligned} L(\lambda_1, \lambda_2) &= \left\{ \prod_{i=1}^{n} \lambda_1 \exp(-\lambda_1 X_i) \right\} \left\{ \prod_{i=1}^{m} \lambda_1 \lambda_2 \exp(-\lambda_1 \lambda_2 Y_i) \right\} \\ &= \lambda_1^n \exp(-\lambda_1 \sum_{i=1}^{n} X_i) \lambda_1^m \lambda_2^m \exp(-\lambda_1 \lambda_2 \sum_{i=1}^{m} Y_i) \end{aligned}$$

and so the log–likelihood function is:

$$l(\lambda_1, \lambda_2) = n \log \lambda_1 - \lambda_1 \sum_{i=1}^{n} X_i + m \log \lambda_1 + m \log \lambda_2 - \lambda_1 \lambda_2 \sum_{i=1}^{m} Y_i.$$

The first derivatives of $l(\lambda_1, \lambda_2)$ are

$$\frac{dl(\lambda_1, \lambda_2)}{d\lambda_1} = \frac{n}{\lambda_1} - \sum_{i=1}^{n} X_i + \frac{m}{\lambda_1} - \lambda_2 \sum_{i=1}^{m} Y_i$$

and

$$\frac{dl(\lambda_1, \lambda_2)}{d\lambda_2} = \frac{m}{\lambda_2} - \lambda_1 \sum_{i=1}^{m} Y_i.$$

Setting these to zero and solving gives the solutions $\hat{\lambda}_1 = n / \sum_{i=1}^{n} X_i = 1/\bar{X}$ and $\hat{\lambda}_2 = \bar{X}/\bar{Y}$.

7. The likelihood function of δ is

$$\begin{aligned} L(\delta) &= \begin{cases} \prod_{i=1}^{n} \exp(\delta - X_i), & \text{if all } X_i \geq \delta, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \exp(n\delta) \exp(-\sum_{i=1}^{n} X_i), & \text{if } \delta \leq \min(X_1, X_2, \ldots, X_n), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\exp(n\delta) \exp(-\sum_{i=1}^{n} X_i)$ in a monotonically increasing function of δ over the range $-\infty < \delta \leq \min(X_1, X_2, \ldots, X_n)$. So, the maximum possible value for $L(\delta)$ will be attained when $\delta = \min(X_1, X_2, \ldots, X_n)$. Hence, the mle of δ is $\hat{\delta} = \min(X_1, X_2, \ldots, X_n)$.