MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 3

1. We have $E(\hat{p}) = E(X/n) = np/n = p$ and so $\hat{p}$ is unbiased. Also $Var(\hat{p}) = Var(X/n) = (1/n^2)Var(X) = (1/n^2)np(1-p) = p(1-p)/n$. So, $MSE(\hat{p}) = p(1-p)/n$ and $\hat{p}$ is MSE consistent since $p(1-p)/n \to 0$ as $n \to \infty$.

2. We have $E(\sum_{i=1}^n X_i^2/n) = (1/n)\sum_{i=1}^n E(X_i^2) = (1/n)\sum_{i=1}^n \sigma^2 = \sigma^2$ and so $\sum_{i=1}^n X_i^2/n$ is an unbiased estimator of σ^2.

   Since $\sum_{i=1}^n X_i^2/\sigma^2 \sim \chi_n^2$, we have $Var(\sum_{i=1}^n X_i^2/\sigma^2) = 2n$ which implies $Var(\sum_{i=1}^n X_i^2/n) = 2\sigma^4/n$ and this approaches zero as $n \to \infty$. Hence, $\sum_{i=1}^n X_i^2/n$ is MSE consistent for σ^2.

3. The estimator $\bar{X}$ is a biased for δ since

$$
\begin{aligned}
E(\bar{X}) &= (1/n)E(\sum_{i=1}^n X_i) \\
&= (1/n)\sum_{i=1}^n E(X_i) \\
&= E(X) \\
&= \int_\delta^\infty x\exp(\delta - x)dx \\
&= \exp(\delta)\int_\delta^\infty x\exp(-x)dx \\
&= \exp(\delta)\left\{[-x\exp(-x)]_\delta^\infty + \int_\delta^\infty \exp(-x)dx\right\} \\
&= \exp(\delta)\left\{\delta\exp(-\delta) + [-\exp(-x)]_\delta^\infty\right\} \\
&= \exp(\delta)\left\{\delta\exp(-\delta) + \exp(-\delta)\right\} \\
&= \delta + 1.
\end{aligned}
$$

   Define a new estimator $\hat{\delta}_2 = \bar{X} - 1$. Since $E(\hat{\delta}_2) = E(\bar{X} - 1) = \delta + 1 - 1 = \delta$ it is unbiased for δ. To check for consistency, first note that

$$
\begin{aligned}
E(X^2) &= \int_\delta^\infty x^2\exp(\delta - x)dx \\
&= \exp(\delta)\int_\delta^\infty x^2\exp(-x)dx \\
&= \exp(\delta)\left\{\left[-x^2\exp(-x)\right]_\delta^\infty + 2\int_\delta^\infty x\exp(-x)dx\right\} \\
&= \exp(\delta)\left\{\delta^2\exp(-\delta) + 2\delta\exp(-\delta) + 2\exp(-\delta)\right\} \\
&= \delta^2 + 2\delta + 2
\end{aligned}
$$

   and

$$
\begin{aligned}
Var(X) &= \delta^2 + 2\delta + 2 - (\delta + 1)^2 \\
&= \delta^2 + 2\delta + 2 - \delta^2 - 2\delta - 1 \\
&= 1.
\end{aligned}
$$

So, $Var(\hat{\delta}_2) = Var(\bar{X} - 1) = Var(\bar{X}) = (1/n)Var(X) = 1/n$ and $MSE(\hat{\delta}_2) = 1/n$. Hence, $\hat{\delta}_2$ is MSE consistent for δ.

4. If $X_1, X_2, \ldots, X_n$ are iid from $Po(\lambda)$ then the mle of λ is $\hat{\lambda} = \bar{X} = (1/n)\sum_{i=1}^n X_i$. Since $E(X_i) = \lambda$ and $Var(X_i) = \lambda$, we have $E(\hat{\lambda}) = (1/n)\sum_{i=1}^n E(X_i) = (1/n)\sum_{i=1}^n \lambda = n\lambda/n = \lambda$ and $Var(\hat{\lambda}) = (1/n^2)\sum_{i=1}^n Var(X_i) = (1/n^2)\sum_{i=1}^n \lambda = n\lambda/n^2 = \lambda/n$. Hence, $MSE(\hat{\lambda}) = \lambda/n$ and $\hat{\lambda}$ is a consistent estimator for λ.

5. If $X_1, X_2, \ldots, X_n$ is a random sample from the geometric distribution with parameter p then the likelihood function for p is:

$$\begin{aligned} L(p) &= \prod_{i=1}^n (1-p)^{X_i - 1} p \\ &= (1-p)^{\sum_{i=1}^n (X_i - 1)} p^n \\ &= (1-p)^{\sum_{i=1}^n X_i - n} p^n \end{aligned}$$

and so the log–likelihood function is:

$$l(p) = (\sum_{i=1}^n X_i - n)\log(1-p) + n\log p.$$

The first derivative of $l(p)$ is

$$\frac{dl(p)}{dp} = -\frac{\sum_{i=1}^n X_i - n}{1-p} + \frac{n}{p}$$

and setting this to zero gives the solution $\hat{p} = n/\sum_{i=1}^n x_i = 1/\bar{X}$. This is indeed the mle since the second derivative

$$\frac{d^2 l(p)}{dp^2} = -\frac{\sum_{i=1}^n x_i - n}{(1-p)^2} - \frac{n}{p^2} < 0.$$

6. The likelihood function of p is

$$L(p) = \left[p^0(1-p)^2\right]^{n_0} \left[2p(1-p)\right]^{n_1} \left[p^2(1-p)^0\right]^{n_2}$$

and so the log–likelihood function is

$$l(p) = n_1 \log 2 + (2n_0 + n_1)\log(1-p) + (n_1 + 2n_2)\log p.$$

The first derivative of $l(p)$ is

$$\frac{dl(p)}{dp} = \frac{n_1 + 2n_2}{p} - \frac{2n_0 + n_1}{1-p}$$

and setting this to zero gives the solution $\hat{p} = (n_1 + 2n_2)/2N$. This is indeed the mle since the second derivative

$$\frac{d^2 l(p)}{dp^2} = -\frac{n_1 + 2n_2}{p^2} - \frac{2n_0 + n_1}{(1-p)^2} < 0.$$

7. Let p denote the proportion of the breakfast cereal Cocobix bought by men.

(i) The likelihood function is

$$L(p) \quad = \quad p^{58}(1-p)^{12}.$$

(ii) The log–likelihood function is

$$l(p) \quad = \quad 58 \log p + 12 \log(1-p).$$

The first derivative of $l(p)$ is

$$\frac{dl(p)}{dp} \quad = \quad \frac{58}{p} - \frac{12}{1-p}$$

and setting this to zero gives the solution $\hat{p} = 29/35$. This is indeed the mle since the second derivative

$$\frac{d^2 l(p)}{dp^2} \quad = \quad -\frac{58}{p^2} - \frac{12}{(1-p)^2} < 0.$$

An approximate 95% confidence interval for p is $[\hat{p} \pm z_{0.025}\sqrt{\hat{p}(1-\hat{p})/n}] \equiv [29/35 \pm 1.96\sqrt{(29/35)(1-29/35)/70}] \equiv [0.74, 0.92]$.

(iii) From the figure below, it is clear that the maximum value of $L(p)$ in the range $1/2 \leq p \leq 2/3$ is at $p = 2/3$. So, the mle is $\hat{p} = 2/3$.