MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 1

1. The mgf does not exist since

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= 3\int_1^\infty \exp(tx)x^{-4}dx \\
&> 3\int_1^\infty \frac{t^4x^4}{4!}x^{-4}dx \\
&\qquad \left[\text{since } e^y > \tfrac{y^k}{k!} \text{ for } y > 0 \text{ and any } k = 0, 1, \ldots\right] \\
&= \frac{t^4}{8}\int_1^\infty 1 \cdot dx \\
&= \frac{t^4}{8}\left[x\right]_1^\infty \\
&= \frac{t^4}{8}\left[\infty - 1\right] \\
&= \infty.
\end{aligned}$$

2. The mgf does not exist since

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_0^\infty \exp(tx)x^{-3}\exp\left(-x^{-1}\right)dx \\
&> \int_0^\infty \frac{t^3x^3}{3!}x^{-3}\exp\left(-x^{-1}\right)dx \\
&\qquad \left[\text{since } e^y > \tfrac{y^k}{k!} \text{ for } y > 0 \text{ and any } k = 0, 1, \ldots\right] \\
&= \frac{t^3}{6}\int_0^\infty \exp\left(-x^{-1}\right)dx \\
&= -\frac{t^3}{6}\int_\infty^0 y^{-2}\exp\left(-y\right)dy \\
&\qquad \left[\text{setting } y = 1/x\right] \\
&= \frac{t^3}{6}\int_0^\infty y^{-2}\exp\left(-y\right)dy \\
&= \frac{t^3}{6}\Gamma(-1)
\end{aligned}$$

$$= \infty.$$

3. The mgf does not exist since

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= 2\int_0^\infty \exp(tx)(1+x)^{-3}dx \\
&> 2\int_0^\infty \frac{t^2x^2}{2!}(1+x)^{-3}dx \\
&\qquad \text{[since } e^y > \tfrac{y^k}{k!} \text{ for } y>0 \text{ and any } k=0,1,\ldots] \\
&= t^2\int_0^\infty x^2(1+x)^{-3}dx \\
&= t^2\int_1^\infty (y-1)^2y^{-3}dy \\
&\qquad \text{setting } y=1+x \\
&= t^2\int_1^\infty \left(y^2-2y+1\right)y^{-3}dy \\
&= t^2\int_1^\infty \left(y^{-1}-2y^{-2}+y^{-3}\right)dy \\
&= t^2\left[\log y+2y^{-1}-\frac{1}{2}y^{-2}\right]_1^\infty \\
&= t^2\left[\infty-0\right] \\
&= \infty.
\end{aligned}$$

4. The mgf does not exist since

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp(tx)x^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)dx \\
&> \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{t^kx^k}{k!}x^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)dx \\
&\qquad \text{[since } e^y > \tfrac{y^k}{k!} \text{ for } y>0 \text{ and any } k=0,1,\ldots] \\
&= \frac{t^k}{k!\sqrt{2\pi}}\int_0^\infty x^{k-1}\exp\left(-\frac{(\log x)^2}{2}\right)dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{t^k}{k!\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[ky\right] \exp\left(-\frac{y^2}{2}\right) dy \\
&\qquad \text{[setting } y = \log x\text{]} \\
&= \frac{t^k}{k!\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[ky - \frac{y^2}{2}\right] dy \\
&= \frac{t^k}{k!\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[y^2 - 2ky\right]\right\} dy \\
&= \frac{t^k}{k!\sqrt{2\pi}} \exp\left[\frac{k^2}{2}\right] \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(y-k)^2\right\} dy \\
&= \frac{t^k}{k!\sqrt{2\pi}} \exp\left[\frac{k^2}{2}\right] \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(y-k)^2\right\} dy \\
&= \frac{t^k}{k!} \exp\left[\frac{k^2}{2}\right] \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(y-k)^2\right\} dy \\
&= \frac{t^k}{k!} \exp\left[\frac{k^2}{2}\right] \\
&\qquad \text{[since } \tfrac{1}{\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}(y-k)^2\right\} dy \text{ is a normal pdf]} \\
&\to \infty
\end{aligned}
$$

as $k \to \infty$.

5. The mgf does not exist since

$$
\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_0^{\infty} \exp(tx) x^{-2} \exp\left(-x^{-1}\right) dx \\
&> \int_0^{\infty} \frac{t^2 x^2}{2!} x^{-2} \exp\left(-x^{-1}\right) dx \\
&\qquad \text{[since } e^y > \tfrac{y^k}{k!} \text{ for } y > 0 \text{ and any } k = 0, 1, \ldots\text{]} \\
&= \frac{t^2}{2} \int_0^{\infty} \exp\left(-x^{-1}\right) dx \\
&= -\frac{t^2}{2} \int_{\infty}^{0} y^{-2} \exp(-y)\, dy \\
&\qquad \text{[setting } y = 1/x\text{]} \\
&= \frac{t^2}{2} \int_0^{\infty} y^{-2} \exp(-y)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{t^2}{2}\Gamma(-1) \\
&= \infty.
\end{aligned}
$$