MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 2

1. We have

$$\begin{aligned} E(\hat{\mu}_1) &= E\left[(X_1 + 2X_2 + X_3)/4\right] \\ &= (1/4)\left[E(X_1) + 2E(X_2) + E(X_3)\right] \\ &= (1/4)\left[\mu + 2\mu + \mu\right] \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} E(\hat{\mu}_2) &= E\left[(X_1 + X_2 + X_3)/3\right] \\ &= (1/3)\left[E(X_1) + E(X_2) + E(X_3)\right] \\ &= (1/3)\left[\mu + \mu + \mu\right] \\ &= \mu, \end{aligned}$$

so both estimators are unbiased. The variances are:

$$\begin{aligned} Var(\hat{\mu}_1) &= Var\left[(X_1 + 2X_2 + X_3)/4\right] \\ &= (1/16)\left[Var(X_1) + 4Var(X_2) + Var(X_3)\right] \\ &= (1/16)\left[\sigma^2 + 4\sigma^2 + \sigma^2\right] \\ &= 3\sigma^2/8 \end{aligned}$$

and

$$\begin{aligned} Var(\hat{\mu}_2) &= Var\left[(X_1 + X_2 + X_3)/3\right] \\ &= (1/9)\left[Var(X_1) + Var(X_2) + Var(X_3)\right] \\ &= (1/9)\left[\sigma^2 + \sigma^2 + \sigma^2\right] \\ &= \sigma^2/3, \end{aligned}$$

so since $Var(\hat{\mu}_2) < Var(\hat{\mu}_1)$ it is best to choose the estimator $\hat{\mu}_2$.

2. (i) We have

$$\begin{aligned} E\left(\sum_{i=1}^n a_i X_i\right) &= \sum_{i=1}^n a_i E(X_i) \\ &= \sum_{i=1}^n a_i \mu \\ &= \mu \sum_{i=1}^n a_i \\ &= \mu \end{aligned}$$

if and only if $\sum_{i=1}^n a_i = 1$.

(ii) We have

$$
\begin{aligned}
Var(\sum_{i=1}^{n} a_i X_i) &= \sum_{i=1}^{n} a_i^2 Var(X_i) \\
&= \sum_{i=1}^{n} a_i^2 \sigma^2 \\
&= \sigma^2 \sum_{i=1}^{n} a_i^2 \\
&= \sigma^2 \sum_{i=1}^{n} \left(a_i - \frac{1}{n} + \frac{1}{n} \right)^2 \\
&= \sigma^2 \sum_{i=1}^{n} \left\{ \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \left(a_i - \frac{1}{n} \right) + \frac{1}{n^2} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right) + \sum_{i=1}^{n} \frac{1}{n^2} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right) + \frac{1}{n} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \left(\left(\sum_{i=1}^{n} a_i \right) - \left(\sum_{i=1}^{n} \frac{1}{n} \right) \right) + \frac{1}{n} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \left(\left(\sum_{i=1}^{n} a_i \right) - 1 \right) + \frac{1}{n} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \left(\sum_{i=1}^{n} a_i \right) - \frac{2}{n} + \frac{1}{n} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{2}{n} \left(\sum_{i=1}^{n} a_i \right) - \frac{1}{n} \right\} \\
&= \sigma^2 \left\{ \sum_{i=1}^{n} \left(a_i - \frac{1}{n} \right)^2 + \frac{1}{n} \right\},
\end{aligned}
$$

where the last step follows since $\sum_{i=1}^{n} a_i = 1$. Minimizing $Var(\sum_{i=1}^{n} a_i X_i)$ amounts to minimizing $\sum_{i=1}^{n}(a_i - 1/n)^2$ and the latter will take the minimum possible value of 0 if $a_i = 1/n$ for all i.

3. Since $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ we have $E((n-1)S^2/\sigma^2) = n-1$ (using the fact that if $X \sim \chi^2_\nu$ then $E(X) = \nu$) and so $E(S^2) = \sigma^2$. Using the fact that if $X \sim \chi^2_\nu$ then $Var(X) = 2\nu$, $Var((n-1)S^2/\sigma^2) = 2(n-1)$ and so $Var(S^2) = 2\sigma^4/(n-1)$. S^2 is a MSE consistent estimator of σ^2 since $MSE(S^2) = 2\sigma^4/(n-1) \to 0$ as $n \to \infty$.

4. First note by substituting $y = x^2/(2\gamma)$ that

$$
\begin{aligned}
E(X^2) &= (1/\gamma) \int_0^\infty x^3 \exp(-x^2/(2\gamma)) dx \\
&= 2\gamma \int_0^\infty y \exp(-y) dy \\
&= 2\gamma \Gamma(2) \\
&= 2\gamma,
\end{aligned}
$$

$$
\begin{aligned}
E(X^4) &= (1/\gamma)\int_0^\infty x^5 \exp(-x^2/(2\gamma))dx \\
&= 4\gamma^2 \int_0^\infty y^2 \exp(-y)dy \\
&= 4\gamma^2 \Gamma(3) \\
&= 8\gamma^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Var(X^2) &= 8\gamma^2 - (2\gamma)^2 \\
&= 4\gamma^2.
\end{aligned}
$$

(i) We have

$$
\begin{aligned}
E(\hat{\gamma}) &= E\left[(1/(2n))\sum_{i=1}^n X_i^2\right] \\
&= (1/(2n))\sum_{i=1}^n E(X_i^2) \\
&= (1/(2n))\sum_{i=1}^n 2\gamma \\
&= \gamma
\end{aligned}
$$

and so $\hat{\gamma}$ is unbiased.

(ii) We have

$$
\begin{aligned}
Var(\hat{\gamma}) &= Var\left[(1/(2n))\sum_{i=1}^n X_i^2\right] \\
&= (1/(4n^2))\sum_{i=1}^n Var(X_i^2) \\
&= (1/(4n^2))\sum_{i=1}^n 4\gamma^2 \\
&= \gamma^2/n
\end{aligned}
$$

and so $MSE(\hat{\gamma}) = \gamma^2/n$.

(iii) By the central limit theorem, $\hat{\gamma} \sim N(\gamma, \gamma^2/n)$.