MATH20802: STATISTICAL METHODS
SEMESTER 2
SOLUTIONS TO PROBLEM SHEET 0

1. If $X \sim N(\mu, \sigma^2)$ then

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^{\infty} \exp(tx) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + tx\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2\mu x - 2\sigma^2 tx + \mu^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 + \mu^2 - (\mu+\sigma^2 t)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma t)^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).
\end{aligned}$$

2. Let $Y = aX_1 + bX_2 + c$. The mgf of Y is

$$\begin{aligned}
M_Y(t) &= E\left[\exp(tY)\right] \\
&= E\left[\exp(taX_1 + tbX_2 + tc)\right] \\
&= \exp(tc) E\left[\exp(taX_1)\right] E\left[\exp(tbX_2)\right] \\
&= \exp(tc) \exp\left(a\mu_1 t + \frac{a^2\sigma_1^2 t^2}{2}\right) \exp\left(b\mu_2 t + \frac{b^2\sigma_2^2 t^2}{2}\right) \\
&= \exp\left((a\mu_1 + b\mu_2 + c)t + \frac{(a^2\sigma_1^2 + b^2\sigma_2^2)t^2}{2}\right),
\end{aligned}$$

which is the mgf of $N(a\mu_1 + b\mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$.

3. Let $Y = \bar{X}$. The mgf of Y is

$$\begin{aligned}
M_Y(t) &= E\left[\exp(tY)\right] \\
&= \prod_{i=1}^{n} E\left[\exp(tX_i/n)\right] \\
&= \prod_{i=1}^{n} \exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right) \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right),
\end{aligned}$$

which is the mgf of $N(\mu, \sigma^2/n)$.

4. If $X \sim Uni(a,b)$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \int_a^b \frac{\exp(tx)}{b-a} dx \\ &= \frac{1}{b-a}\int_a^b \exp(tx)dx \\ &= \frac{1}{t(b-a)}\left[\exp(tx)\right]_a^b \\ &= \frac{\exp(bt)-\exp(at)}{t(b-a)} \end{aligned}$$

as required.

5. If $X \sim Exp(\lambda)$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \lambda \int_0^\infty \exp(tx - \lambda x)dx \\ &= \frac{\lambda}{\lambda - t}\left[\exp(-(\lambda - t)x)\right]_0^\infty \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

provided that $\lambda - t > 0$.

6. If $X \sim Ga(a, \lambda)$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \frac{\lambda^a}{\Gamma(a)}\int_0^\infty x^{a-1}\exp(tx - \lambda x)dx \\ &= \frac{\lambda^a}{(\lambda - t)^a\Gamma(a)}\int_0^\infty y^{a-1}\exp(-y)dy \\ &= \frac{\lambda^a}{(\lambda - t)^a} \end{aligned}$$

provided that $\lambda - t > 0$ (we have set $y = (\lambda - t)x$).