MATH20802: STATISTICAL METHODS
SEMESTER 2
PROBLEM SHEET 2

1. If X_1, X_2, X_3 constitute a random sample from a $N(\mu, \sigma^2)$ distribution with μ unknown then consider the following estimators of μ:

$$\hat{\mu}_1 = (X_1 + 2X_2 + X_3)/4$$

   and

$$\hat{\mu}_2 = (X_1 + X_2 + X_3)/3.$$

   Which of these two estimators would it be best to use in practice and why?

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2.

   (i) Show that $\sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ for any set of known constants $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^n a_i = 1$.

   (ii) If $\sum_{i=1}^n a_i = 1$ show that $Var(\sum_{i=1}^n a_i X_i)$ is minimized for $a_i = 1/n$, $i = 1, 2, \ldots, n$ (Hint: prove that $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n (a_i - 1/n)^2 + 1/n$ when $\sum_{i=1}^n a_i = 1$.)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Find $E(S^2)$ and $Var(S^2)$ where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

   If S^2 a MSE consistent estimator of σ^2? (Hint: Recall that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.)

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution

$$f(x) = \begin{cases} (x/\gamma)\exp(-x^2/(2\gamma)), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   (i) Show that $\hat{\gamma} = (1/(2n)) \sum_{i=1}^n X_i^2$ is an unbiased estimator of γ.

   (ii) Fine the MSE of $\hat{\gamma}$.

   (iii) What is the approximate distribution of $\hat{\gamma}$ for large n.