SOLUTIONS TO
STATISTICAL METHODS EXAM

Solutions to Question 1 A random X is said to have the hyperbolic secant distribution if its probability density function is given by

$$f_X(x) = \frac{1}{\exp\left(\frac{\pi x}{2}\right) + \exp\left(-\frac{\pi x}{2}\right)}$$

for $-\infty < x < +\infty$.

(a) The cumulative distribution function of X is

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{1}{\exp\left(\frac{\pi y}{2}\right) + \exp\left(-\frac{\pi y}{2}\right)} dy \\
&= -\frac{2}{\pi} \int_{+\infty}^{\exp\left(-\frac{\pi x}{2}\right)} \frac{1}{1+z^2} dz \\
&= \frac{2}{\pi} \int_{\exp\left(-\frac{\pi x}{2}\right)}^{+\infty} \frac{1}{1+z^2} dz \\
&= \frac{2}{\pi} \left[\arctan z\right]_{\exp\left(-\frac{\pi x}{2}\right)}^{+\infty} \\
&= \frac{2}{\pi} \left[\frac{\pi}{2} - \arctan \exp\left(-\frac{\pi x}{2}\right)\right] \\
&= 1 - \frac{2}{\pi} \arctan \exp\left(-\frac{\pi x}{2}\right).
\end{aligned}$$

where we have set $z = \exp\left(-\frac{\pi y}{2}\right)$. (6 marks)

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(b) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^{+\infty} \frac{\exp(tx)}{\exp\left(\frac{\pi x}{2}\right) + \exp\left(-\frac{\pi x}{2}\right)} dx \\
&= -\frac{2}{\pi} \int_{+\infty}^{0} \frac{z^{-\frac{2t}{\pi}}}{1+z^2} dz \\
&= \frac{2}{\pi} \int_{0}^{+\infty} \frac{z^{-\frac{2t}{\pi}}}{1+z^2} dz \\
&= -\frac{1}{\pi} \int_{1}^{0} y^{\frac{t}{\pi}-\frac{1}{2}} (1-y)^{-\frac{t}{\pi}-\frac{1}{2}} dy \\
&= \frac{1}{\pi} \int_{0}^{1} y^{\frac{t}{\pi}-\frac{1}{2}} (1-y)^{-\frac{t}{\pi}-\frac{1}{2}} dy \\
&= \frac{1}{\pi} B\left(\frac{t}{\pi} + \frac{1}{2}, -\frac{t}{\pi} + \frac{1}{2}\right) \\
&= \frac{1}{\pi} \frac{\Gamma\left(\frac{t}{\pi} + \frac{1}{2}\right) \Gamma\left(-\frac{t}{\pi} + \frac{1}{2}\right)}{1} \\
&= \frac{1}{\pi} \Gamma\left(\frac{t}{\pi} + \frac{1}{2}\right) \Gamma\left(-\frac{t}{\pi} + \frac{1}{2}\right).
\end{aligned}$$

where we have set $z = \exp\left(-\frac{\pi x}{2}\right)$ and $y = \frac{1}{1+z^2}$. (6 marks)

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(c) The first two derivatives of the mgf are

$$M_X^{'}(t) = \frac{1}{\pi^2}\Gamma^{'}\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma\left(-\frac{t}{\pi}+\frac{1}{2}\right) - \frac{1}{\pi^2}\Gamma\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma^{'}\left(-\frac{t}{\pi}+\frac{1}{2}\right)$$

and

$$\begin{aligned} M_X^{''}(t) &= \frac{1}{\pi^3}\Gamma^{''}\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma\left(-\frac{t}{\pi}+\frac{1}{2}\right) - \frac{1}{\pi^3}\Gamma^{'}\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma^{'}\left(-\frac{t}{\pi}+\frac{1}{2}\right) \\ &\quad -\frac{1}{\pi^3}\Gamma^{'}\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma^{'}\left(-\frac{t}{\pi}+\frac{1}{2}\right) + \frac{1}{\pi^3}\Gamma\left(\frac{t}{\pi}+\frac{1}{2}\right)\Gamma^{''}\left(-\frac{t}{\pi}+\frac{1}{2}\right) \end{aligned}$$

So,

$$E(X) = M_X^{'}(0) = \frac{1}{\pi^2}\Gamma^{'}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) - \frac{1}{\pi^2}\Gamma\left(\frac{1}{2}\right)\Gamma^{'}\left(\frac{1}{2}\right) = 0$$

and

$$E\left(X^2\right) = M_X^{''}(0) = \frac{2}{\pi^3}\Gamma^{''}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) - \frac{2}{\pi^3}\Gamma^{'}\left(\frac{1}{2}\right)\Gamma^{'}\left(\frac{1}{2}\right).$$

(6 marks)

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(d) The cumulative distribution function of $\mid Y \mid$ is

$$\begin{aligned} \Pr\left(\mid Y \mid < y\right) &= \Pr\left(\exp\left(\pi X/2\right) < y\right) \\ &= \Pr\left(X < \frac{2}{\pi}\log y\right) \\ &= F_X\left(\frac{2}{\pi}\log y\right) \\ &= 1 - \frac{2}{\pi}\arctan\exp\left(-\log y\right) \\ &= 1 - \frac{2}{\pi}\arctan\frac{1}{y}. \end{aligned}$$

Differentiating with respect to y, the probability density function of $\mid Y \mid$ is

$$\frac{2}{\pi}\frac{1}{1+y^2}.$$

Since this is symmetric around zero, the probability density and cumulative distribution functions of Y are

$$f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2}$$

and

$$F_Y(y) = \frac{1}{\pi}\arctan y - \frac{1}{2},$$

respectively. (5 marks)

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(e) Cauchy distribution. (2 marks)

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Solutions to Question 2 (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ if $\lim_{n\to\infty} E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(iii) the bias of $\widehat{\theta}$ is $E\left(\widehat{\theta}\right) - \theta$; (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ is $E\left[\left(\widehat{\theta} - \theta\right)^2\right]$; (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E\left[\left(\widehat{\theta} - \theta\right)^2\right] = 0$. (2 marks)

UP TO THIS BOOK WORK.

(b) Suppose X_1 and X_2 are independent Exp$(1/\theta)$ and Uniform $[0, \theta]$ random variables. Let $\widehat{\theta} = aX_1 + bX_2$ denote a class of estimators of θ, where a and b are constants.

(i) The bias of $\widehat{\theta}$ is

$$
\begin{aligned}
\text{Bias}\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}\right) - \theta \\
&= aE\left(X_1\right) + bE\left(X_2\right) - \theta \\
&= \frac{a}{\theta}\int_0^{+\infty} x \exp\left(-\frac{x}{\theta}\right) dx + b\int_0^{\theta} \frac{x}{\theta} dx - \theta \\
&= a\theta \int_0^{+\infty} y \exp\left(-y\right) dx + \frac{b}{\theta}\left[\frac{x^2}{2}\right]_0^{\theta} - \theta \\
&= a\theta \int_0^{+\infty} y \exp\left(-y\right) dx + \frac{b}{\theta}\left[\frac{\theta^2}{2} - 0\right] - \theta \\
&= \left(a + \frac{b}{2} - 1\right)\theta.
\end{aligned}
$$

(3 marks)

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(ii) The variance of $\widehat{\theta}$ is

$$
\begin{aligned}
\text{Var}\left(\widehat{\theta}\right) &= a^2 \text{Var}\left(X_1\right) + b^2 \text{Var}\left(X_2\right) \\
&= a^2\left[\frac{1}{\theta}\int_0^{+\infty} x^2 \exp\left(-\frac{x}{\theta}\right) dx - \theta^2\right] + b^2\left[\frac{1}{\theta}\int_0^{\theta} x^2 dx - \frac{\theta^2}{4}\right] \\
&= a^2\left[\theta^2 \int_0^{+\infty} y^2 \exp\left(-y\right) dy - \theta^2\right] + b^2\left\{\frac{1}{\theta}\left[\frac{x^3}{3}\right]_0^{\theta} - \frac{\theta^2}{4}\right\} \\
&= a^2\left[2\theta^2 - \theta^2\right] + b^2\left\{\frac{\theta^2}{3} - \frac{\theta^2}{4}\right\} \\
&= \left(a^2 + \frac{b^2}{12}\right)\theta^2.
\end{aligned}
$$

(3 marks)

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(iii) The mean squared error of $\widehat{\theta}$ is

$$\begin{aligned}\mathrm{MSE}\left(\widehat{\theta}\right) &= \left(a^2+\frac{b^2}{12}\right)\theta^2+\left(a+\frac{b}{2}-1\right)^2\theta^2 \\ &= \left[a^2+\frac{b^2}{12}+\left(a+\frac{b}{2}-1\right)^2\right]\theta^2 \\ &= \left(2a^2+\frac{b^2}{3}+ab-2a-b+1\right)\theta^2.\end{aligned}$$

(2 marks)

(iv) $\widehat{\theta}$ is unbiased if $a+\frac{b}{2}=1$. In other words, $b=2(1-a)$. (2 marks)

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(v) If $\widehat{\theta}$ is unbiased then its variance is

$$\left[a^2+\frac{(1-a)^2}{3}\right]\theta^2.$$

We need to minimize this as a function of a. Let $g(a)=a^2+\frac{(1-a)^2}{3}$. The first order derivative is $g^{'}(a)=2a-\frac{2(1-a)}{3}$. Setting the derivative to zero, we obtain $a=\frac{1}{4}$. The second order derivative is $g^{''}(a)=2+\frac{2}{3}>0$. So, $g(a)=a^2+\frac{(1-a)^2}{3}$ attains its minimum at $a=\frac{1}{4}$. Hence, the estimator with minimum variance is $\frac{1}{4}X_1+\frac{3}{2}X_2$. (5 marks)

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Solutions to Question 3 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x > 0$.

(a) The likelihood function of σ^2 is

$$\begin{aligned} L\left(\sigma^2\right) &= \prod_{i=1}^{n} \left[\frac{X_i}{\sigma^2} \exp\left(-\frac{X_i^2}{2\sigma^2}\right)\right] \\ &= \frac{1}{\sigma^{2n}} \left(\prod_{i=1}^{n} X_i\right) \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i^2\right). \end{aligned}$$

(4 marks)

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(b) The log likelihood function of σ^2 is

$$\log L\left(\sigma^2\right) = -2n \log \sigma + \prod_{i=1}^{n} \log X_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i^2.$$

The derivative with respect to σ is

$$\frac{d \log L\left(\sigma^2\right)}{d\sigma} = -\frac{2n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} X_i^2.$$

Setting this to zero gives

$$\widehat{\sigma^2} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2.$$

This is a maximum likelihood estimator since

$$\begin{aligned} \frac{d^2 \log L\left(\sigma^2\right)}{d\sigma^2} &= \frac{2n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} X_i^2 \\ &= \frac{1}{\sigma^4} \left[2n\sigma^2 - 3 \sum_{i=1}^{n} X_i^2\right] \\ &= \frac{1}{\sigma^4} \left[2n \frac{1}{2n} \sum_{i=1}^{n} X_i^2 - 3 \sum_{i=1}^{n} X_i^2\right] \\ &< 0 \end{aligned}$$

at $\sigma = \widehat{\sigma}$. (4 marks)

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(c) By the invariance principle, the maximum likelihood estimator of σ is

$$\widehat{\sigma} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2}.$$

(1 mark)

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(d) The bias of $\widehat{\sigma^2}$ is

$$
\begin{aligned}
\mathrm{Bias}\left(\widehat{\sigma^2}\right) &= E\left(\widehat{\sigma^2}\right) - \sigma^2 \\
&= E\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) - \sigma^2 \\
&= \frac{1}{2n}\sum_{i=1}^{n} E\left(X_i^2\right) - \sigma^2 \\
&= \frac{1}{2n\sigma^2}\sum_{i=1}^{n} \int_0^\infty x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n} \int_0^\infty y \exp(-y)\, dy - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n} \Gamma(2) - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n} 1 - \sigma^2 \\
&= 0.
\end{aligned}
$$

Hence, $\widehat{\sigma^2}$ is unbiased for σ^2. (8 marks)

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(e) The mean squared error of $\widehat{\sigma^2}$ is

$$
\begin{aligned}
\mathrm{MSE}\left(\widehat{\sigma^2}\right) &= \mathrm{Var}\left(\widehat{\sigma^2}\right) \\
&= \mathrm{Var}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \mathrm{Var}\left(X_i^2\right) \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{E\left(X_i^4\right) - \left[E\left(X_i^2\right)\right]^2\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{E\left(X_i^4\right) - \left[2\sigma^2\right]^2\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{4\sigma^4 \int_0^\infty y^2 \exp(-y)\, dy - 4\sigma^4\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{4\sigma^4 \Gamma(3) - 4\sigma^4\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{8\sigma^4 - 4\sigma^4\right\} \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

Hence, $\widehat{\sigma^2}$ is consistent σ^2. (8 marks)

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Solutions to Question 4 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f_X(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$$

for $x > 0$, $\mu > 0$ and $\lambda > 0$. Assume both μ and λ are unknown.

(a) The joint likelihood function of λ and μ is

$$\begin{aligned}
L(\lambda, \mu) &= \prod_{i=1}^{n} \left\{ \sqrt{\frac{\lambda}{2\pi X_i^3}} \exp\left[-\frac{\lambda\left(X_i - \mu\right)^2}{2\mu^2 X_i}\right] \right\} \\
&= \frac{\lambda^{n/2}}{(2\pi)^{n/2}} \left(\prod_{i=1}^{n} X_i\right)^{-3/2} \exp\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^{n} \frac{\left(X_i - \mu\right)^2}{X_i}\right\} \\
&= \frac{\lambda^{n/2}}{(2\pi)^{n/2}} \left(\prod_{i=1}^{n} X_i\right)^{-3/2} \exp\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^{n} \frac{X_i^2 - 2\mu X_i + \mu^2}{X_i}\right\} \\
&= \frac{\lambda^{n/2}}{(2\pi)^{n/2}} \left(\prod_{i=1}^{n} X_i\right)^{-3/2} \exp\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^{n} \left(X_i - 2\mu + \frac{\mu^2}{X_i}\right)\right\} \\
&= \frac{\lambda^{n/2}}{(2\pi)^{n/2}} \left(\prod_{i=1}^{n} X_i\right)^{-3/2} \exp\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^{n} X_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2} \sum_{i=1}^{n} \frac{1}{X_i}\right\}.
\end{aligned}$$

(5 marks)

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(b) The joint log-likelihood function is

$$\log L(\lambda, \mu) = \frac{n}{2} \log \lambda - \frac{n}{2} \log(2\pi) - \frac{3}{2} \sum_{i=1}^{n} \log X_i - \frac{\lambda}{2\mu^2} \sum_{i=1}^{n} X_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2} \sum_{i=1}^{n}.$$

The partial derivative with respect to μ is

$$\frac{\partial \log L(\lambda, \mu)}{\partial \mu} = \frac{\lambda}{\mu^3} \sum_{i=1}^{n} X_i - \frac{n\lambda}{\mu^2}.$$

Setting this to zero and solving for μ, we obtain

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}.$$

(5 marks)

UNSEEN

(c) The partial derivative of the log-likelihood function with respect to λ is

$$\frac{\partial \log L(\lambda, \mu)}{\partial \lambda} = \frac{n}{2\lambda} - \frac{1}{2\mu^2} \sum_{i=1}^{n} X_i + \frac{n}{\mu} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{X_i}.$$

Setting this to zero and replying μ by $\overline{X}$, we see

$$\frac{n}{2\lambda} - \frac{n^2}{2}\left(\sum_{i=1}^{n} X_i\right)^{-1} + \frac{n}{\overline{X}} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{X_i} = 0$$

or equivalently

$$\frac{n}{2\lambda} + \frac{n^2}{2}\left(\sum_{i=1}^{n} X_i\right)^{-1} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{X_i} = 0.$$

So, the solution for λ is

$$\widehat{\lambda} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i} - \frac{1}{\overline{X}}\right)^{-1}.$$

(5 marks)

UNSEEN

(d) The bias and mean squared error of $\widehat{\mu}$ are

$$\begin{aligned}
\text{Bias}\left(\widehat{\mu}\right) &= E\left(\widehat{\mu}\right) - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu - \mu \\
&= \mu - \mu \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}\left(\widehat{\mu}\right) &= \text{Var}\left(\widehat{\mu}\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}\left(X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \frac{\mu^3}{\lambda} \\
&= \frac{\mu^3}{n\lambda}.
\end{aligned}$$

Hence, $\widehat{\mu}$ is unbiased and consistent. (5 marks)

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(e) By the hint,

$$\frac{1}{\widehat{\lambda}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i} - \frac{1}{\overline{X}} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{X_i} - \frac{n}{\overline{X}}\right) \sim \frac{1}{\lambda n}\cdot\chi^2_{n-1}.$$

So,

$$
\begin{aligned}
\text{Bias}\left(\frac{1}{\widehat{\lambda}}\right) &= E\left(\frac{1}{\widehat{\lambda}}\right) - \frac{1}{\lambda} \\
&= \frac{1}{n\lambda} E\left(\chi^2_{n-1}\right) - \frac{1}{\lambda} \\
&= \frac{n-1}{n\lambda} - \frac{1}{\lambda} \\
&= -\frac{1}{n\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{MSE}\left(\frac{1}{\widehat{\lambda}}\right) &= \text{Var}\left(\frac{1}{\widehat{\lambda}}\right) + \left[\text{Bias}\left(\frac{1}{\widehat{\lambda}}\right)\right]^2 \\
&= \text{Var}\left(\frac{1}{\lambda n}\chi^2_{n-1}\right) + \left[-\frac{1}{n\lambda}\right]^2 \\
&= \frac{1}{\lambda^2 n^2}\text{Var}\left(\chi^2_{n-1}\right) + \frac{1}{n^2\lambda^2} \\
&= \frac{2(n-1)}{\lambda^2 n^2} + \frac{1}{n^2\lambda^2}
\end{aligned}
$$

Hence, $\widehat{\lambda}$ is unbiased and consistent. (5 marks)

UNSEEN

Solutions to Question 5 (a) Suppose we wish to test $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\mu = \mu_0$; (2 marks)
SEEN

(ii) the Type II error occurs if H_0 is accepted when in fact $\mu \neq \mu_0$; (2 marks)
SEEN

(iii) the significance level is the probability of type I error; (2 marks)
SEEN

(iv) the power function: $\Pi(\mu) = \Pr\left(\text{Reject} H_0 \mid \mu\right)$. (2 marks)
SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, \sigma^2\right)$, where σ is not known.

(i) The rejection region for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$ is

$$\frac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2}.$$

(2 marks)

SEEN

(ii) The rejection region for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma < \sigma_0$ is

$$\frac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha}.$$

(2 marks)

SEEN

(iii) The rejection region for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$ is

$$\frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha}.$$

(2 marks)

SEEN

In each case, we have assumed a significance level of α.

(c) Under the same assumptions as in part (b), the power function, $\Pi(\sigma)$, for each of the tests is as follows.

(i) The power function, $\Pi(\sigma)$, for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$ is

$$\begin{aligned}
\Pi(\sigma) &= \Pr\left(\frac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha/2} \text{ or } \frac{(n-1)S^2}{\sigma^2} > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha/2} \middle| \sigma\right) \\
&= \Pr\left(\chi^2_{n-1} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha/2} \text{ or } \chi^2_{n-1} > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha/2} \middle| \sigma\right) \\
&= \Pr\left(\chi^2_{n-1} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha/2}\right) + 1 - \Pr\left(\chi^2_{n-1} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha/2}\right) \\
&= F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha/2}\right) + 1 - F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha/2}\right).
\end{aligned}$$

(5 marks)

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(ii) The power function, $\Pi(\sigma)$, for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma < \sigma_0$ is

$$\begin{aligned}
\Pi(\sigma) &= \Pr\left(\frac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha} \middle| \sigma\right) \\
&= \Pr\left(\chi^2_{n-1} < \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha} \middle| \sigma\right) \\
&= F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,1-\alpha}\right).
\end{aligned}$$

(3 marks)

UNSEEN

(iii) The power function, $\Pi(\sigma)$, for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$ is

$$\begin{aligned}
\Pi(\sigma) &= \Pr\left(\frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{\sigma_0^2} > \chi^2_{n-1,\alpha} \middle| \sigma\right) \\
&= \Pr\left(\frac{(n-1)S^2}{\sigma^2} > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha} \middle| \sigma\right) \\
&= \Pr\left(\chi^2_{n-1} > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha} \middle| \sigma\right)
\end{aligned}$$

$$= \quad 1 - F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{n-1,\alpha}\right).$$

(3 marks)

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Note that we have used the fact $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Furthermore, $F_{\chi^2_{n-1}}$ denotes the cumulative distribution function of χ^2_{n-1}.

Solutions to Question 6

(a) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ if

$$\frac{L(\theta_1)}{L(\theta_2)} = \frac{\prod_{i=1}^{n} f(X_i; \theta_1)}{\prod_{i=1}^{n} f(X_i; \theta_2)} < k$$

for some k. (4 marks)

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UP TO THIS BOOK WORK.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Exp (θ).

(i) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, $\theta_2 > \theta_1$ if

$$\begin{aligned}
\frac{L(\theta_1)}{L(\theta_2)} &= \frac{\prod_{i=1}^{n} \left[\theta_1 \exp(-\theta_1 X_i)\right]}{\prod_{i=1}^{n} \left[\theta_2 \exp(-\theta_2 X_i)\right]} \\
&= \frac{\theta_1^n \exp\left(-\theta_1 \sum_{i=1}^{n} X_i\right)}{\theta_2^n \exp\left(-\theta_2 \sum_{i=1}^{n} X_i\right)} \\
&= \frac{\theta_1^n}{\theta_2^n} \exp\left[(\theta_2 - \theta_1) \sum_{i=1}^{n} X_i\right] \\
&< k,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \exp\left[(\theta_2 - \theta_1) \sum_{i=1}^{n} X_i\right] < \frac{\theta_2^n}{\theta_1^n} k \\
\Leftrightarrow \quad & (\theta_2 - \theta_1) \sum_{i=1}^{n} X_i < \log\left[\frac{\theta_2^n}{\theta_1^n} k\right] \\
\Leftrightarrow \quad & \sum_{i=1}^{n} X_i < \frac{1}{\theta_2 - \theta_1} \log\left[\frac{\theta_2^n}{\theta_1^n} k\right] = c
\end{aligned}$$

say. (8 marks)

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(ii) Let $Z = \sum_{i=1}^{n} X_i$. Suppose $n = 1$. Then the power function is

$$\Pi(\theta) = \Pr(X_1 < c \mid \theta) = \Pr(\mathrm{Exp}(\theta) < c \mid \theta) = 1 - \exp(-\theta c),$$

so the statement holds when $n = 1$. Suppose now $n = 2$. Then the power function is

$$\begin{aligned}
\Pi(\theta) &= \Pr\left(X_1 + X_2 < c \mid \theta\right) \\
&= \Pr\left(\mathrm{Exp}(\theta) + \mathrm{Exp}(\theta) < c \mid \theta\right) \\
&= \int_0^c \left\{1 - \exp\left[-\theta(c-x)\right]\right\} \theta \exp(-\theta x) dx \\
&= \int_0^c \theta \exp(-\theta x) dx - \int_0^c \theta \exp\left(-\theta c\right) dx \\
&= 1 - \exp(-\theta c) - \theta c \exp\left(-\theta c\right) \\
&= 1 - (1 + \theta c) \exp(-\theta c),
\end{aligned}$$

so the statement holds when $n = 2$. Next, assume that the statement is true for $n = k - 1$, that is

$$\Pi(\theta) = 1 - \left[1 + \theta c + \cdots + \frac{\theta^{k-2} c^{k-2}}{(k-2)!}\right] \exp(-\theta c).$$

The statement also holds for $n = k$ since

$$\begin{aligned}
\Pi(\theta) &= \Pr\left(X_1 + \cdots + X_n < c \mid \theta\right) \\
&= \int_0^c \left\{1 - \left[1 + \theta(c-x) + \cdots + \frac{\theta^{k-2}(c-x)^{k-2}}{(k-2)!}\right] \exp\left[-\theta(c-x)\right]\right\} \theta \exp(-\theta x) dx \\
&= \int_0^c \theta \exp(-\theta x) dx - \theta \exp(-\theta c) \int_0^c \left[1 + \theta(c-x) + \cdots + \frac{\theta^{k-2}(c-x)^{k-2}}{(k-2)!}\right] dx \\
&= \int_0^c \theta \exp(-\theta x) dx - \theta \exp(-\theta c) \left[x - \theta \frac{(c-x)^2}{2} - \cdots - \frac{\theta^{k-2}(c-x)^{k-1}}{(k-1)!}\right]_0^c \\
&= 1 - \exp(-\theta c) - \theta \exp(-\theta c) \left[c + \theta \frac{c^2}{2} + \cdots + \frac{\theta^{k-2} c^{k-1}}{(k-1)!}\right] \\
&= 1 - \exp(-\theta c) - \exp(-\theta c) \left[\theta c + \frac{\theta^2 c^2}{2} + \cdots + \frac{\theta^{k-1} c^{k-1}}{(k-1)!}\right] \\
&= 1 - \left[1 + \theta c + \frac{\theta^2 c^2}{2} + \cdots + \frac{\theta^{k-1} c^{k-1}}{(k-1)!}\right] \exp(-\theta c).
\end{aligned}$$

Hence, the result follows. (8 marks)

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(iii) If $n = 2$, $\theta_1 = 1$ and $\alpha = 0.05$ then

$$1 - (1 + c) \exp(-c) = 0.05.$$

Compute the right hand side when $c = 0.3553595$. It will become equal to 0.05. (2 marks)

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(iv) If $n = 2$, $\theta_1 = 1$, $\theta_2 = 2$ and $\alpha = 0.05$ then

$$\Pr\left(\mathrm{TypeIIerror}\right) = \Pr\left(\mathrm{Accept} H_0 \mid H_1 \mathrm{istrue}\right)$$

$$
\begin{aligned}
&= 1 - \Pr\left(\text{Reject} H_0 \mid H_1 \text{is true}\right) \\
&= 1 - [1 - (1+2c)\exp(-2c)] \\
&= (1+2c)\exp(-2c) \\
&= 0.8404606.
\end{aligned}
$$

(3 marks)

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