SOLUTIONS TO
STATISTICAL METHODS EXAM

Solutions to Question 1 Let X denote a random variable with its probability density function given by

$$f_X(x) = \frac{\exp\left(-\dfrac{x-\mu}{s}\right)}{s\left[1+\exp\left(-\dfrac{x-\mu}{s}\right)\right]^2}$$

for $-\infty < x < +\infty$, $-\infty < \mu < +\infty$ and $s > 0$. X is said to have a logistic distribution with parameters μ and s.

(a) The cumulative distribution function of X is

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{\exp\left(-\dfrac{y-\mu}{s}\right)}{s\left[1+\exp\left(-\dfrac{y-\mu}{s}\right)\right]^2} dy \\
&= -\int_{+\infty}^{\exp\left(-\frac{x-\mu}{s}\right)} \frac{1}{[1+z]^2} dz \\
&= \int_{\exp\left(-\frac{x-\mu}{s}\right)}^{+\infty} \frac{1}{[1+z]^2} dz \\
&= \left\{-[1+z]^{-1}\right\}_{\exp\left(-\frac{x-\mu}{s}\right)}^{+\infty} \\
&= 0 - \left\{-\left[1+\exp\left(-\frac{x-\mu}{s}\right)\right]^{-1}\right\} \\
&= \left[1+\exp\left(-\frac{x-\mu}{s}\right)\right]^{-1},
\end{aligned}$$

where we have set $z = \exp\left(-\frac{y-\mu}{s}\right)$. (6 marks)

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(b) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^{+\infty} \frac{\exp\left(tx - \dfrac{x-\mu}{s}\right)}{s\left[1+\exp\left(-\dfrac{x-\mu}{s}\right)\right]^2} dx \\
&= -\int_{+\infty}^{0} \frac{\exp\left[t\left(\mu - s\log z\right)\right]}{[1+z]^2} dz \\
&= \exp(\mu t)\int_{0}^{+\infty} \frac{z^{-st}}{[1+z]^2} dz \\
&= -\exp(\mu t)\int_{1}^{0} \left(\frac{1-w}{w}\right)^{-st} dw \\
&= \exp(\mu t)\int_{0}^{1} w^{st}(1-w)^{-st} dw \\
&= \exp(\mu t)\, B(1-st, 1+st),
\end{aligned}$$

where we have set $z = \exp\left(-\frac{y-\mu}{s}\right)$ and $w = \frac{1}{1+z}$. (6 marks)

UNSEEN

(c) The mgf can be rewritten as

$$M_X(t) = \exp(\mu t)\,\Gamma(1-st)\,\Gamma(1+st).$$

Its first two derivatives are

$$\begin{aligned}
M_X^{'}(t) &= \mu\exp(\mu t)\,\Gamma(1-st)\,\Gamma(1+st) - s\exp(\mu t)\,\Gamma^{'}(1-st)\,\Gamma(1+st)\\
&\quad +s\exp(\mu t)\,\Gamma(1-st)\,\Gamma^{'}(1+st),\\
M_X^{''}(t) &= \mu^2\exp(\mu t)\,\Gamma(1-st)\,\Gamma(1+st) - \mu s\exp(\mu t)\,\Gamma^{'}(1-st)\,\Gamma(1+st)\\
&\quad +\mu s\exp(\mu t)\,\Gamma(1-st)\,\Gamma^{'}(1+st) - \mu s\exp(\mu t)\,\Gamma^{'}(1-st)\,\Gamma(1+st)\\
&\quad +s^2\exp(\mu t)\,\Gamma^{''}(1-st)\,\Gamma(1+st) - s^2\exp(\mu t)\,\Gamma^{'}(1-st)\,\Gamma^{'}(1+st)\\
&\quad +\mu s\exp(\mu t)\,\Gamma(1-st)\,\Gamma^{'}(1+st) - s^2\exp(\mu t)\,\Gamma^{'}(1-st)\,\Gamma^{'}(1+st)\\
&\quad +s^2\exp(\mu t)\,\Gamma(1-st)\,\Gamma^{''}(1+st).
\end{aligned}$$

Setting $t=0$ gives

$$\begin{aligned}
E(X) &= M_X^{'}(0) = \mu\Gamma(1)\,\Gamma(1) - s\Gamma^{'}(1)\,\Gamma(1) + s\Gamma(1)\,\Gamma^{'}(1) = \mu,\\
E\left(X^2\right) &= M_X^{''}(0) = \mu^2\Gamma(1)\,\Gamma(1) - \mu s\Gamma^{'}(1)\,\Gamma(1) + \mu s\Gamma(1)\,\Gamma^{'}(1)\\
&\quad -\mu s\Gamma^{'}(1)\,\Gamma(1) + s^2\Gamma^{''}(1)\,\Gamma(1) - s^2\Gamma^{'}(1)\,\Gamma^{'}(1)\\
&\quad +\mu s\Gamma(1)\,\Gamma^{'}(1) - s^2\Gamma^{'}(1)\,\Gamma^{'}(1) + s^2\Gamma(1)\,\Gamma^{''}() = \mu^2 +\\
&\quad 2s^2\Gamma^{''}(1) - 2s^2\left[\Gamma^{'}(1)\right]^2.
\end{aligned}$$

So, the variance of X is $2s^2\Gamma^{''}(1) - 2s^2\left[\Gamma^{'}(1)\right]^2$. (6 marks)

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(d) Suppose Y and Z are independent unit exponential random variables. The moment generating function of $W = -\log\left(\frac{Y}{Z}\right) = -\log Y + \log Z$ is

$$\begin{aligned}
M_W(t) &= E\left[\exp(tW)\right]\\
&= E\left[\exp(-t\log Y + t\log Z)\right]\\
&= E\left[\exp(-t\log Y)\right] E\left[\exp(t\log Z)\right]\\
&= \left[\int_0^\infty \exp(-t\log y)\exp(-y)dy\right]\left[\int_0^\infty \exp(t\log z)\exp(-z)dz\right]\\
&= \left[\int_0^\infty y^{-t}\exp(-y)dy\right]\left[\int_0^\infty z^{t}\exp(-z)dz\right]\\
&= \Gamma(1-t)\Gamma(1+t).
\end{aligned}$$

(5 marks)

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(e) Since $M_W(t) = \Gamma(1-t)\Gamma(1+t) = B(1-t, 1+t)$, W follows the logistic distribution with $\mu = 0$ and $s = 1$. (2 marks)

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Solutions to Question 2 (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ if $\lim_{n\to\infty} E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(iii) the bias of $\widehat{\theta}$ is $E\left(\widehat{\theta}\right) - \theta$; (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ is $E\left[\left(\widehat{\theta} - \theta\right)^2\right]$; (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E\left[\left(\widehat{\theta} - \theta\right)^2\right] = 0$. (2 marks)

UP TO THIS BOOK WORK.

(b) Suppose X_1 and X_2 are independent Exp$(1/\lambda)$ random variables. Let $\widehat{\theta_1} = a\left(X_1 + X_2\right)$ and $\widehat{\theta_2} = b\sqrt{X_1 X_2}$ denote possible estimators of λ, where a and b are constants.

(i) The expectation of $\widehat{\theta_1}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_1}\right) &= a\left[E\left(X_1\right) + E\left(X_2\right)\right] \\
&= 2a \int_0^{+\infty} \frac{x}{\lambda} \exp\left(-\frac{x}{\lambda}\right) dx \\
&= 2a\lambda \int_0^{+\infty} y \exp\left(-y\right) dy \\
&= 2a\lambda.
\end{aligned}
$$

So, $\widehat{\theta_1}$ is unbiased for λ if $a = 1/2$. (3 marks)

UNSEEN

(ii) The expectation of $\widehat{\theta_2}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_2}\right) &= bE\left(\sqrt{X_1}\right) E\left(\sqrt{X_2}\right) \\
&= b\left[\int_0^{+\infty} \frac{\sqrt{x}}{\lambda} \exp\left(-\frac{x}{\lambda}\right) dx\right]^2 \\
&= b\left[\int_0^{+\infty} \sqrt{\lambda y} \exp\left(-y\right) dy\right]^2 \\
&= b\lambda \left[\int_0^{+\infty} \sqrt{y} \exp\left(-y\right) dy\right]^2 \\
&= b\lambda \left[\Gamma(3/2)\right]^2 \\
&= b\lambda\pi/4.
\end{aligned}
$$

So, $\widehat{\theta_2}$ is unbiased for λ if $b = 4/\pi$. (3 marks)

UNSEEN

(iii) The variance of $\widehat{\theta}_1$ is

$$\begin{aligned}
\mathrm{Var}\left(\widehat{\theta}_1\right) &= a^2\left[\mathrm{Var}\left(X_1\right)+\mathrm{Var}\left(X_2\right)\right] \\
&= \\
&= 2a^2\mathrm{Var}\left(X_1\right) \\
&= 2a^2\left[\int_0^{+\infty}\frac{x^2}{\lambda}\exp\left(-\frac{x}{\lambda}\right)dx-\lambda^2\right] \\
&= 2a^2\left[\lambda^2\int_0^{+\infty}y^2\exp\left(-y\right)dy-\lambda^2\right] \\
&= 2a^2\left[\lambda^2\Gamma(3)-\lambda^2\right] \\
&= 2a^2\lambda^2 \\
&= \lambda^2/2.
\end{aligned}$$

(3 marks)

UNSEEN

(iv) The variance of $\widehat{\theta}_2$ is

$$\mathrm{Var}\left(\widehat{\theta}_2\right)=b^2E\left(X_1\right)E\left(X_2\right)-\lambda^2=b^2\lambda^2-\lambda^2=\left(b^2-1\right)\lambda^2=\left(\frac{16}{\pi^2}-1\right)\lambda^2.$$

(3 marks)

UNSEEN

(v) Clearly, $\lambda^2/2<\left(\frac{16}{\pi^2}-1\right)\lambda^2$, so the estimator $\widehat{\theta}_1$ is better with respect to mean squared error.

(3 marks)

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Solutions to Question 3 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the discrete uniform distribution on the set of integers $\{1, 2, \ldots, N\}$, where $N \geq 1$ is an unknown parameter.

(a) The likelihood function of N is

$$\begin{aligned} L(N) &= \prod_{i=1}^{n} \left[\frac{1}{N} I\{1 \leq X_i \leq N\} \right] \\ &= \frac{1}{N^n} \prod_{i=1}^{n} [I\{1 \leq X_i \leq N\}] \\ &= \frac{1}{N^n} I\{1 \leq \min(X_1, \ldots, X_N)\} \, I\{\max(X_1, \ldots, X_N) \leq N\}. \end{aligned}$$

(3 marks)

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(b) Plot $L(N)$ versus N. You will see $L(N)$ attains its largest value when $N = \max(X_1, X_2, \ldots, X_N)$. Hence, the maximum likelihood estimator of N is $\max(X_1, X_2, \ldots, X_N)$. (3 marks)

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(c) Let $Z = \max(X_1, X_2, \ldots, X_N)$. The cumulative distribution function of Z is

$$\begin{aligned} \Pr(Z \leq z) &= \Pr[\max(X_1, X_2, \ldots, X_N) \leq z] \\ &= \Pr[X_1 \leq z, X_2 \leq z, \ldots, X_N \leq z] \\ &= \Pr[X_1 \leq z] \Pr[X_2 \leq z] \cdots \Pr[X_N \leq z] \\ &= \frac{z}{N} \frac{z}{N} \cdots \frac{z}{N} \\ &= \left(\frac{z}{N}\right)^n \end{aligned}$$

for integer z. The probability mass function of Z is

$$\left(\frac{z}{N}\right)^n - \left(\frac{z-1}{N}\right)^n$$

for integer z. (3 marks)

UNSEEN

(d) $\max(X_1, X_2, \ldots, X_N)$ is a biased estimator of N since

$$\begin{aligned} & \mathrm{Bias}[\max(X_1, X_2, \ldots, X_N)] \\ =\ & E[\max(X_1, X_2, \ldots, X_N)] - N \\ =\ & \sum_{z=1}^{N} z \left[\left(\frac{z}{N}\right)^n - \left(\frac{z-1}{N}\right)^n \right] - N \\ =\ & \frac{1}{N^n} \sum_{z=1}^{N} \left[z^{n+1} - z(z-1)^n \right] - N \\ =\ & \frac{1}{N^n} \sum_{z=1}^{N} \left[z^{n+1} - (z-1+1)(z-1)^n \right] - N \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N^n}\sum_{z=1}^{N}\left[z^{n+1}-(z-1)^{n+1}-(z-1)^n\right]-N \\
&= \frac{1}{N^n}\sum_{z=1}^{N}\left[z^{n+1}-(z-1)^{n+1}\right]-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n-N \\
&= \frac{1}{N^n}N^{n+1}-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n-N \\
&= -\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n.
\end{aligned}
$$

$\max\left(X_1, X_2, \ldots, X_N\right)$ is an asymptotically unbiased estimator of N since

$$
-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n=-\sum_{z=1}^{N}\left(\frac{z-1}{N}\right)^n \to 0
$$

as $n \to \infty$. (8 marks)

UNSEEN

(e) $\max\left(X_1, X_2, \ldots, X_N\right)$ is a consistent estimator of N since

$$
\begin{aligned}
& \operatorname{Var}\left[\max\left(X_1, X_2, \ldots, X_N\right)\right] \\
= \ & E\left[\left(\max\left(X_1, X_2, \ldots, X_N\right)\right)^2\right]-E^2\left[\max\left(X_1, X_2, \ldots, X_N\right)\right] \\
= \ & \sum_{z=1}^{N} z^2\left[\left(\frac{z}{N}\right)^n-\left(\frac{z-1}{N}\right)^n\right]-\left(N-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & \frac{1}{N^n}\sum_{z=1}^{N} z^2\left[z^n-(z-1)^n\right]-\left(N-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & \frac{1}{N^n}\sum_{z=1}^{N}\left[z^{n+2}-(z-1+1)^2(z-1)^n\right]-\left(N-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & \frac{1}{N^n}\sum_{z=1}^{N}\left[z^{n+2}-(z-1)^{n+2}\right]-\frac{1}{N^n}\sum_{z=1}^{N}\left[2(z-1)^{n+1}+(z-1)^n\right]-\left(N-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & N^2-\frac{1}{N^n}\sum_{z=1}^{N}\left[2(z-1)^{n+1}+(z-1)^n\right]-\left(N-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & -\frac{1}{N^n}\sum_{z=1}^{N}\left[2(z-1)^{n+1}+(z-1)^n\right]+\frac{2N}{N^n}\sum_{z=1}^{N}(z-1)^n-\left(\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & \frac{2}{N^n}\sum_{z=1}^{N}(z-1)^n(N-z+1)-\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n-\left(\frac{1}{N^n}\sum_{z=1}^{N}(z-1)^n\right)^2 \\
= \ & 2\sum_{z=1}^{N}\left(\frac{z-1}{N}\right)^n(N-z+1)-\sum_{z=1}^{N}\left(\frac{z-1}{N}\right)^n-\left(\sum_{z=1}^{N}\left(\frac{z-1}{N}\right)^n\right)^2 \\
\to \ & 0
\end{aligned}
$$

as $n \to \infty$. (8 marks)

UNSEEN

Solutions to Question 4 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f_X(x) = \frac{1}{s} \exp\left(-\frac{x-\mu}{s}\right)$$

for $x > \mu$, $\mu > 0$ and $s > 0$. Assume both μ and s are unknown.

(a) The joint likelihood function of μ and s is

$$\begin{aligned} L(\mu, s) &= \prod_{i=1}^{n} \left[\frac{1}{s} \exp\left(-\frac{X_i - \mu}{s}\right) I\left\{X_i \geq \mu\right\}\right] \\ &= \frac{1}{s^n} \prod_{i=1}^{n} \left[\exp\left(-\frac{X_i - \mu}{s}\right) I\left\{X_i \geq \mu\right\}\right] \\ &= \frac{1}{s^n} \exp\left(-\frac{1}{s} \sum_{i=1}^{n} X_i + \frac{n\mu}{s}\right) \left[\prod_{i=1}^{n} I\left\{X_i \geq \mu\right\}\right] \\ &= \frac{1}{s^n} \exp\left(-\frac{1}{s} \sum_{i=1}^{n} X_i + \frac{n\mu}{s}\right) I\left\{\min\left(X_1, \ldots, X_n\right) \geq \mu\right\} \end{aligned}$$

(3 marks)

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(b) Plot $L(\mu, s)$ versus μ for fixed s. You will see L is a monotonic increasing function over $(0, \min(X_1, X_2, \ldots, X_n)]$ and zero thereafter. So, it attains its largest value when $\mu = \min(X_1, X_2, \ldots, X_n)$. Hence, the maximum likelihood estimator of μ is $\min(X_1, X_2, \ldots, X_n)$.

(3 marks)

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(c) The log likelihood function of s for fixed μ is

$$\log L(s) = -n \log s - \frac{1}{s} \sum_{i=1}^{n} X_i + \frac{n\mu}{s} + \log I\left\{\min\left(X_1, \ldots, X_n\right) \geq \mu\right\}$$

The derivative with respect to s is

$$\frac{\partial \log L(s)}{\partial s} = -\frac{n}{s} + \frac{1}{s^2} \sum_{i=1}^{n} X_i - \frac{n\mu}{s^2}.$$

Setting to zero and solving for s, we obtain

$$\widehat{s} = \frac{1}{n}\left[\sum_{i=1}^{n} X_i - n\widehat{\mu}\right] = \left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) - \min\left(X_1, X_2, \ldots, X_n\right).$$

(3 marks)

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(d) Let $Z = \min(X_1, X_2, \ldots, X_n)$. The cdf of Z is

$$\begin{aligned}
F_Z(z) &= \Pr[\min(X_1, X_2, \ldots, X_n) < z] \\
&= 1 - \Pr[\min(X_1, X_2, \ldots, X_n) \geq z] \\
&= 1 - \Pr[X_1 \geq z, X_2 \geq z, \ldots, X_n \geq z] \\
&= 1 - \Pr[X_1 \geq z] \Pr[X_2 \geq z] \cdots \Pr[X_n \geq z] \\
&= 1 - (\Pr[X_1 \geq z])^n \\
&= 1 - \left(\frac{1}{s} \int_z^{+\infty} \exp\left(-\frac{x-\mu}{s} \right) dx \right)^n \\
&= 1 - \left(\left[-\exp\left(-\frac{x-\mu}{s} \right) \right]_z^{+\infty} \right)^n \\
&= 1 - \exp\left(-n \frac{z-\mu}{s} \right).
\end{aligned}$$

The corresponding pdf is

$$f_Z(z) = \frac{n}{s} \exp\left(-n \frac{z-\mu}{s} \right).$$

The expectation of Z is

$$\begin{aligned}
E(Z) &= \frac{n}{s} \int_\mu^{+\infty} z \exp\left(-n \frac{z-\mu}{s} \right) dz \\
&= \frac{n}{s} \int_\mu^{+\infty} (z - \mu + \mu) \exp\left(-n \frac{z-\mu}{s} \right) dz \\
&= n \int_\mu^{+\infty} \frac{z-\mu}{s} \exp\left(-n \frac{z-\mu}{s} \right) dz + \frac{n\mu}{s} \int_\mu^{+\infty} \exp\left(-n \frac{z-\mu}{s} \right) dz \\
&= ns \int_0^{+\infty} w \exp(-nw)\, dw + n\mu \int_0^{+\infty} \exp(-nw)\, dw \\
&= \frac{s}{n} \int_0^{+\infty} w \exp(-w)\, dw + \mu \int_0^{+\infty} \exp(-w)\, dw \\
&= \mu + \frac{s}{n}.
\end{aligned}$$

The bias of Z is $\mu + \frac{s}{n} - \mu = \frac{s}{n} \neq 0$, so Z is biased for μ. (8 marks)

UNSEEN

(e) The expectation of $\widehat{s}$ is

$$\begin{aligned}
E(\widehat{s}) &= \left(\frac{1}{n} \sum_{i=1}^n E(X_i) \right) - E[\min(X_1, X_2, \ldots, X_n)] \\
&= \frac{1}{s} \int_\mu^{+\infty} z \exp\left(-\frac{z-\mu}{s} \right) dz - \mu - \frac{s}{n} \\
&= \frac{1}{s} \int_\mu^{+\infty} (z - \mu + \mu) \exp\left(-\frac{z-\mu}{s} \right) dz - \mu - \frac{s}{n} \\
&= \int_\mu^{+\infty} \frac{z-\mu}{s} \exp\left(-\frac{z-\mu}{s} \right) dz + \frac{\mu}{s} \int_\mu^{+\infty} \exp\left(-\frac{z-\mu}{s} \right) dz - \mu - \frac{s}{n}
\end{aligned}$$

$$
\begin{aligned}
&= s\int_0^{+\infty} w\exp(-w)\,dw + \mu\int_0^{+\infty}\exp(-w)\,dw - \mu - \frac{s}{n} \\
&= s + \mu - \mu - \frac{s}{n} \\
&= s - \frac{s}{n}.
\end{aligned}
$$

The bias of $\widehat{s}$ is $s - \frac{s}{n} - s \neq 0$, so $\widehat{s}$ is biased for s. (8 marks)

UNSEEN

Solutions to Question 5 (a) Suppose we wish to test $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\mu = \mu_0$; (2 marks)

(ii) the Type II error occurs if H_0 is accepted when in fact $\mu \neq \mu_0$; (2 marks)

(iii) the significance level is the probability of type I error; (2 marks)

(iv) the power function: $\Pi(\mu) = \Pr(\text{Reject} H_0 \mid \mu)$. (2 marks)

SEEN

(b) Let $X_1, X_2, \ldots, X_m$ be a random sample from a normal population with mean μ_X assumed unknown and variance σ_X^2 assumed known. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal population with mean μ_Y assumed unknown and variance σ_Y^2 assumed known. Assume independence of the two samples. State the rejection region for each of the following tests:

(i) The rejection region for $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$ is

$$\frac{\mid \overline{X} - \overline{Y} \mid}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha/2}.$$

(2 marks)

SEEN

(ii) The rejection region for $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$ is

$$\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha}.$$

(2 marks)

SEEN

(iii) The rejection region for $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X < \mu_Y$ is

$$\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \leq -z_{\alpha}.$$

(2 marks)

SEEN

UP TO THIS BOOK WORK

(c) Let $X_1, X_2, \ldots, X_m$ be a random sample from a normal population with mean μ_X and variance σ_X^2 assumed known. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal population with mean μ_Y and variance σ_Y^2 assumed known. Assume independence of the two samples.

(i) The power function, $\Pi(\mu_X, \mu_Y)$, for testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$ is

$$
\begin{aligned}
\Pi(\mu_X, \mu_Y) &= \Pr\left(\frac{|\,\overline{X} - \overline{Y}\,|}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha/2} \,\middle|\, \mu_X, \mu_Y \right) \\
&= \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha/2} \,\middle|\, \mu_X, \mu_Y \right) + \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \leq -z_{\alpha/2} \,\middle|\, \mu_X, \mu_Y \right) \\
&= \Pr\left(\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&\quad + \Pr\left(\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \leq -z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&= \Pr\left(N(0,1) \geq z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&\quad + \Pr\left(N(0,1) \leq -z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&= 1 - \Phi\left(z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \right) + \Phi\left(-z_{\alpha/2} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \right).
\end{aligned}
$$

(5 marks)

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(ii) The power function, $\Pi(\mu_X, \mu_Y)$, for testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$ is

$$
\begin{aligned}
\Pi(\mu_X, \mu_Y) &= \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha} \,\middle|\, \mu_X, \mu_Y \right) \\
&= \Pr\left(\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \geq z_{\alpha} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&= \Pr\left(N(0,1) \geq z_{\alpha} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X, \mu_Y \right) \\
&= 1 - \Phi\left(z_{\alpha} - \frac{\mu_X - \mu_Y}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \right).
\end{aligned}
$$

(3 marks)

UNSEEN

(iii) The power function, $\Pi(\mu_X, \mu_Y)$, for testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X < \mu_Y$ is

$$
\Pi(\mu_X, \mu_Y) = \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}}} \leq -z_{\alpha} \,\middle|\, \mu_X, \mu_Y \right)
$$

$$
\begin{aligned}
&= \Pr\left(\frac{\overline{X}-\overline{Y}-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}} \leq -z_\alpha - \frac{\mu_X-\mu_Y}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X,\mu_Y\right) \\
&= \Pr\left(N(0,1) \leq -z_\alpha - \frac{\mu_X-\mu_Y}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}} \,\middle|\, \mu_X,\mu_Y\right) \\
&= \Phi\left(-z_\alpha - \frac{\mu_X-\mu_Y}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}}\right).
\end{aligned}
$$

(3 marks)

UNSEEN

Note that we have used the fact $\frac{\overline{X}-\overline{Y}-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}} = Z \sim N(0,1)$. Furthermore, $\Phi(\cdot)$ denotes the cumulative distribution function of $N(0,1)$.

Solutions to Question 6

(a) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ if

$$\frac{L(\theta_1)}{L(\theta_2)} = \frac{\prod_{i=1}^{n} f(X_i; \theta_1)}{\prod_{i=1}^{n} f(X_i; \theta_2)} < k$$

for some k. (4 marks)

SEEN

UP TO THIS BOOK WORK.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f(x) = \frac{1}{2\theta} \exp\left(-\frac{\mid x \mid}{\theta}\right)$$

for $-\infty < x < +\infty$ and $\theta > 0$, where θ is unknown.

(i) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, $\theta_1 > \theta_2$ if

$$\begin{aligned}
\frac{L(\theta_1)}{L(\theta_2)} &= \frac{\prod_{i=1}^{n} \frac{1}{2\theta_1} \exp\left(-\frac{\mid X_i \mid}{\theta_1}\right)}{\prod_{i=1}^{n} \frac{1}{2\theta_2} \exp\left(-\frac{\mid X_i \mid}{\theta_2}\right)} \\
&= \frac{\theta_2^n \exp\left(-\frac{1}{\theta_1} \sum_{i=1}^{n} \mid X_i \mid\right)}{\theta_1^n \exp\left(-\frac{1}{\theta_2} \sum_{i=1}^{n} \mid X_i \mid\right)} \\
&= \frac{\theta_2^n}{\theta_1^n} \exp\left[\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) \sum_{i=1}^{n} \mid X_i \mid\right] \\
&< k,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \frac{\theta_2^n}{\theta_1^n} \exp\left[\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) \sum_{i=1}^{n} \mid X_i \mid\right] < k \\
\Leftrightarrow \quad & \exp\left[\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) \sum_{i=1}^{n} \mid X_i \mid\right] < \frac{\theta_1^n}{\theta_2^n} k \\
\Leftrightarrow \quad & \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) \sum_{i=1}^{n} \mid X_i \mid < \log\left(\frac{\theta_1^n}{\theta_2^n} k\right) \\
\Leftrightarrow \quad & \sum_{i=1}^{n} \mid X_i \mid < \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)^{-1} \log\left(\frac{\theta_1^n}{\theta_2^n} k\right) = c
\end{aligned}$$

say. (8 marks)

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(ii) Let $Z = \sum_{i=1}^{n} | X_i |$. The mgf of Z is

$$
\begin{aligned}
M_Z(t) &= E\left[\exp(tZ)\right] \\
&= E\left[\exp\left(t\sum_{i=1}^{n} | X_i |\right)\right] \\
&= \prod_{i=1}^{n} E\left[\exp\left(t \mid X_i \mid\right)\right] \\
&= \prod_{i=1}^{n} \left[\frac{1}{2\theta}\int_{-\infty}^{+\infty} \exp\left(t \mid x \mid - \frac{| x |}{\theta}\right) dx\right] \\
&= \prod_{i=1}^{n} \left[\frac{1}{\theta}\int_{0}^{+\infty} \exp\left(tx - \frac{x}{\theta}\right) dx\right] \\
&= \prod_{i=1}^{n} \left\{\frac{1}{\theta}\left(t - \frac{1}{\theta}\right)^{-1} \left[\exp\left(tx - \frac{x}{\theta}\right)\right]_0^{+\infty}\right\} \\
&= \prod_{i=1}^{n} \left\{\frac{1}{\theta}\left(t - \frac{1}{\theta}\right)^{-1} [0 - 1]\right\} \\
&= \prod_{i=1}^{n} \left\{\frac{1}{\theta}\left(\frac{1}{\theta} - t\right)^{-1}\right\} \\
&= \frac{1}{\theta^n}\left(\frac{1}{\theta} - t\right)^{-n}
\end{aligned}
$$

for $t < 1/\theta$, so Z is a gamma random variable with scale parameter $\frac{1}{\theta}$ and shape parameter n.

The power function for the rejection rule in part (i) is

$$\Pi(\theta) = \Pr\left(Z < c \mid \theta\right) = F_{\frac{1}{\theta},n}(c),$$

where $F_{a,b}(\cdot)$ denotes the cdf of a gamma random variable with scale parameter a and shape parameter b. (8 marks)

UNSEEN

(iii) If $n = 1$, $\theta_1 = 1$ and $\alpha = 0.05$ then

$$F_{1,1}(c) = 0.05$$

which implies

$$1 - \exp(-c) = 0.05$$

which implies

$$c = -\log 0.95 = 0.051.$$

(2 marks)

UNSEEN

(iv) If $n = 1$, $\theta_1 = 1$, $\theta_2 = 0.5$ and $\alpha = 0.05$ then

$$\begin{aligned}
\Pr(\text{TypeIIerror}) &= \Pr(\text{Accept} H_0 \mid H_1 \text{istrue}) \\
&= 1 - \Pr(\text{Reject} H_0 \mid H_1 \text{istrue}) \\
&= 1 - F_{2,1}(-\log 0.95) \\
&= 1 - [1 - \exp(2 \log 0.95)] \\
&= 0.9025.
\end{aligned}$$

(3 marks)

UNSEEN