SOLUTIONS TO
STATISTICAL METHODS EXAM

Solutions to Question 1 Suppose X is a random variable with its probability density function given by

$$f(x) = \alpha\lambda \exp(-\lambda x) + (1-\alpha)\mu \exp(-\mu x)$$

for $x > 0$, $0 < \alpha < 1$, $\lambda > 0$ and $\mu > 0$.

(a) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= \alpha\lambda \int_0^\infty \exp(tx - \lambda x)dx + (1-\alpha)\mu \int_0^\infty \exp(tx - \mu x)dx \\
&= \alpha\lambda \left[\frac{\exp(tx-\lambda x)}{t-\lambda}\right]_0^\infty + (1-\alpha)\mu \left[\frac{\exp(tx-\mu x)}{t-\mu}\right]_0^\infty \\
&= \alpha\lambda \left[0 - \frac{1}{t-\lambda}\right]_0^\infty + (1-\alpha)\mu \left[0 - \frac{1}{t-\mu}\right]_0^\infty \\
&= \alpha\lambda \frac{1}{\lambda - t} + (1-\alpha)\mu \frac{1}{\mu - t}
\end{aligned}$$

provided $t < \lambda$ and $t < \mu$. (8 marks)

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(b) The first four derivatives of $M_X(t)$ are

$$\begin{aligned}
M_X^{'}(t) &= \frac{\alpha\lambda}{(\lambda-t)^2} + \frac{(1-\alpha)\mu}{(\mu-t)^2}, \\
M_X^{''}(t) &= \frac{2\alpha\lambda}{(\lambda-t)^3} + \frac{2(1-\alpha)\mu}{(\mu-t)^3}, \\
M_X^{'''}(t) &= \frac{6\alpha\lambda}{(\lambda-t)^4} + \frac{6(1-\alpha)\mu}{(\mu-t)^4}, \\
M_X^{''''}(t) &= \frac{24\alpha\lambda}{(\lambda-t)^5} + \frac{24(1-\alpha)\mu}{(\mu-t)^5}.
\end{aligned}$$

So, the first four moments are

$$\begin{aligned}
E(X) = M_X^{'}(0) &= \frac{\alpha}{\lambda} + \frac{1-\alpha}{\mu}, \\
E\left(X^2\right) = M_X^{''}(0) &= \frac{2\alpha}{\lambda^2} + \frac{2(1-\alpha)}{\mu^2}, \\
E\left(X^3\right) = M_X^{'''}(0) &= \frac{6\alpha}{\lambda^3} + \frac{6(1-\alpha)}{\mu^3}, \\
E\left(X^4\right) = M_X^{''''}(0) &= \frac{24\alpha}{\lambda^4} + \frac{24(1-\alpha)}{\mu^4}.
\end{aligned}$$

(8 marks)

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(c) The moment generating function of $Y = X_1 + \cdots + X_n$ is

$$M_Y(t) = E\left[\exp(tY)\right]$$

$$
\begin{aligned}
&= E\left[\exp\left(tX_1 + \cdots + tX_n\right)\right] \\
&= E\left[\exp\left(tX_1\right)\right] \cdots E\left[\exp\left(tX_n\right)\right] \\
&= M_{X_1}(t) \cdots M_{X_n}(t) \\
&= \left[\alpha\lambda \frac{1}{\lambda - t} + (1-\alpha)\mu \frac{1}{\mu - t}\right]^n .
\end{aligned}
$$

(3 marks)

UNSEEN

(d) The mean of Y is

$$
\begin{aligned}
E(Y) &= E\left[X_1 + \cdots + X_n\right] \\
&= E\left[X_1\right] + \cdots + E\left[X_n\right] \\
&= nE[X] \\
&= \frac{n\alpha}{\lambda} + \frac{n(1-\alpha)}{\mu}.
\end{aligned}
$$

The variance of Y is

$$
\begin{aligned}
Var(Y) &= Var\left[X_1 + \cdots + X_n\right] \\
&= Var\left[X_1\right] + \cdots + Var\left[X_n\right] \\
&= nVar[X] \\
&= nE\left[X^2\right] - nE^2[X] \\
&= \frac{2n\alpha}{\lambda^2} + \frac{2n(1-\alpha)}{\mu^2} - n\left[\frac{\alpha}{\lambda} + \frac{1-\alpha}{\mu}\right]^2 .
\end{aligned}
$$

(3 marks)

UNSEEN

(e) If $\lambda = \mu$ then The moment generating function of $Y = X_1 + \cdots + X_n$ is

$$
M_Y(t) = \left[\alpha\lambda \frac{1}{\lambda - t} + (1-\alpha)\lambda \frac{1}{\lambda - t}\right]^n = \frac{\lambda^n}{(\lambda - t)^n},
$$

the moment generating of a gamma random variable. So, Y has the gamma distribution with parameters n and λ. (3 marks)

UNSEEN

Solutions to Question 2 (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ if $\lim_{n \to \infty} E\left(\widehat{\theta}\right) = \theta$; (2 marks)

(iii) the bias of $\widehat{\theta}$ is $E\left(\widehat{\theta}\right) - \theta$; (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ is $E\left[\left(\widehat{\theta} - \theta\right)^2\right]$; (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n \to \infty} E\left[\left(\widehat{\theta} - \theta\right)^2\right] = 0$. (2 marks)

UP TO THIS BOOK WORK.

(b) Suppose $X_1, \ldots, X_n$ are independent Uniform$[0, \theta]$ random variables. Let $\widehat{\theta_1} = \frac{2(X_1 + \cdots + X_n)}{n}$ and $\widehat{\theta_2} = \max\left(X_1, \ldots, X_n\right)$ denote possible estimators of θ.

(i) The bias and mean squared error of $\widehat{\theta_1}$ are

$$
\begin{aligned}
\text{bias}\left(\widehat{\theta_1}\right) &= E\left(\widehat{\theta_1}\right) - \theta \\
&= \frac{2}{n} E\left(X_1 + \cdots + X_n\right) - \theta \\
&= \frac{2}{n}\left[E\left(X_1\right) + \cdots + E\left(X_n\right)\right] - \theta \\
&= \frac{2}{n}\left[\frac{\theta}{2} + \cdots + \frac{\theta}{2}\right] - \theta \\
&= \theta - \theta \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\text{MSE}\left(\widehat{\theta_1}\right) &= \text{Var}\left(\widehat{\theta_1}\right) \\
&= \frac{4}{n^2} \text{Var}\left(X_1 + \cdots + X_n\right) \\
&= \frac{4}{n^2}\left[\text{Var}\left(X_1\right) + \cdots + \text{Var}\left(X_n\right)\right] \\
&= \frac{4}{n^2}\left[\frac{\theta^2}{12} + \cdots + \frac{\theta^2}{12}\right] \\
&= \frac{\theta^2}{3n}.
\end{aligned}
$$

(4 marks)

UNSEEN

(ii) Let $Z = \widehat{\theta_2}$. The cdf and the pdf of Z are

$$\begin{aligned}
F_Z(z) &= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z\right) \\
&= \Pr\left(X_1 \leq z, \ldots, X_n \leq z\right) \\
&= \Pr\left(X_1 \leq z\right) \cdots \Pr\left(X_n \leq z\right) \\
&= \frac{z}{\theta} \cdots \frac{z}{\theta} \\
&= \frac{z^n}{\theta^n}
\end{aligned}$$

and

$$f_Z(z) = \frac{n z^{n-1}}{\theta^n}.$$

So, the bias and mean squared error of $\widehat{\theta_2}$ are

$$\begin{aligned}
\text{bias}\left(\widehat{\theta_2}\right) &= E\left(Z\right) - \theta \\
&= \frac{n}{\theta^n} \int_0^\theta z^n dz - \theta \\
&= \frac{n}{\theta^n} \left[\frac{z^{n+1}}{n+1}\right]_0^\theta - \theta \\
&= \frac{n}{\theta^n} \left[\frac{\theta^{n+1}}{n+1} - 0\right] - \theta \\
&= \frac{n\theta}{n+1} - \theta \\
&= -\frac{\theta}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}\left(\widehat{\theta_1}\right) &= \text{Var}\left(Z\right) + \left(-\frac{\theta}{n+1}\right)^2 \\
&= E\left(Z^2\right) - E^2\left(Z\right) + \frac{\theta^2}{(n+1)^2} \\
&= E\left(Z^2\right) - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n}{\theta^n} \int_0^\theta z^{n+1} dz - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n}{\theta^n} \left[\frac{z^{n+2}}{n+2}\right]_0^\theta - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{2\theta^2}{(n+1)(n+2)}.
\end{aligned}$$

(7 marks)

UNSEEN

(iii) $\widehat{\theta_1}$ is better with respect to bias since bias $\widehat{\theta_1} = 0$ and bias $\widehat{\theta_2} \neq 0$. (1 mark)

UNSEEN

(iv) $\widehat{\theta_2}$ is better with respect to mean squared error since

$$\begin{aligned} & \frac{2\theta^2}{(n+1)(n+2)} \leq \frac{\theta^2}{3n} \\ \Leftrightarrow \quad & 6n \leq (n+1)(n+2) \\ \Leftrightarrow \quad & 6n \leq n^2 + 3n + 2 \\ \Leftrightarrow \quad & 0 \leq n^2 - 3n + 2 \\ \Leftrightarrow \quad & 0 \leq (n-1)(n-2). \end{aligned}$$

Both $\widehat{\theta_1}$ and $\widehat{\theta_2}$ have equal mean squared errors when $n = 1, 2$. (3 marks)

UNSEEN

Solutions to Question 3 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $f(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right)$, $-\infty < x < \infty$, where $a > 0$ is an unknown parameter.

(a) The likelihood function of a is

$$L(a) = \prod_{i=1}^{n} \left[\frac{1}{2a} \exp\left(-\frac{\mid X_i \mid}{a} \right) \right] = \frac{1}{(2a)^n} \exp\left(-\frac{1}{a} \sum_{i=1}^{n} \mid X_i \mid \right).$$

(5 marks)

UNSEEN

(b) The log likelihood function of a is

$$\log L(a) = -n \log(2a) - \frac{1}{a} \sum_{i=1}^{n} \mid X_i \mid .$$

The derivative of $\log L$ with respect to a is

$$\frac{d \log L(a)}{da} = -\frac{n}{a} + \frac{1}{a^2} \sum_{i=1}^{n} \mid X_i \mid .$$

Setting this to zero and solving for a, we obtain

$$\widehat{a} = \frac{1}{n} \sum_{i=1}^{n} \mid X_i \mid .$$

This is a maximum likelihood estimator of a since

$$\begin{aligned} \left. \frac{d^2 \log L(a)}{da^2} \right|_{a=\widehat{a}} &= \frac{n}{\widehat{a}^2} - \frac{2}{\widehat{a}^3} \sum_{i=1}^{n} \mid X_i \mid \\ &= \frac{n}{\widehat{a}^2} \left[1 - \frac{2}{n\widehat{a}} \sum_{i=1}^{n} \mid X_i \mid \right] \\ &= \frac{n}{\widehat{a}^2} [1 - 2] \\ &< 0. \end{aligned}$$

(6 marks)

UNSEEN

(c) The expected value of $\widehat{a}$ is

$$\begin{aligned} E(\widehat{a}) &= \frac{1}{n} \sum_{i=1}^{n} E[\mid X_i \mid] \\ &= \frac{1}{2na} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mid x \mid \exp\left(-\frac{\mid x \mid}{a} \right) dx \\ &= \frac{1}{na} \sum_{i=1}^{n} \int_{0}^{\infty} x \exp\left(-\frac{x}{a} \right) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{n}\sum_{i=1}^{n}\int_0^\infty y\exp(-y)\,dy \\
&= \frac{a}{n}\sum_{i=1}^{n}\Gamma(2) \\
&= \frac{a}{n}\sum_{i=1}^{n}1 \\
&= a.
\end{aligned}
$$

(6 marks)

UNSEEN

(d) The variance of $\widehat{a}$ is

$$
\begin{aligned}
\mathrm{Var}(\widehat{a}) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left[\mid X_i \mid\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left\{E\left[X_i^2\right] - E^2\left[\mid X_i \mid\right]\right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left\{\frac{1}{2a}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{\mid x \mid}{a}\right)dx - \frac{1}{4a^2}\left[\int_{-\infty}^{\infty}\mid x \mid \exp\left(-\frac{\mid x \mid}{a}\right)dx\right]^2\right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left\{\frac{1}{a}\int_0^{\infty} x^2\exp\left(-\frac{x}{a}\right)dx - \frac{1}{a^2}\left[\int_0^{\infty} x\exp\left(-\frac{x}{a}\right)dx\right]^2\right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left\{a^2\int_0^{\infty} y^2\exp(-y)\,dy - a^2\left[\int_0^{\infty} y\exp(-y)\,dy\right]^2\right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left\{a^2\Gamma(3) - a^2\left[\Gamma(2)\right]^2\right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}a^2 \\
&= \frac{a^2}{n}.
\end{aligned}
$$

(6 marks)

UNSEEN

(e) The bias $(\widehat{a}) = 0$ and MSE $(\widehat{a}) = \frac{a^2}{n}$, so $\widehat{a}$ is an unbiased and a consistent estimator for a. (2 marks)

UNSEEN

Solutions to Question 4 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$. Suppose $Y_1, Y_2, \ldots, Y_m$ is a random sample from $LN(\mu, \sigma^2)$ independent of $X_1, X_2, \ldots, X_n$. Assume both μ and σ^2 are unknown.

(a) The joint likelihood function of μ and σ^2 is

$$\begin{aligned}
L\left(\mu, \sigma^2\right) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(X_i-\mu\right)^2}{2\sigma^2} \right] \right\} \prod_{i=1}^{m} \left\{ \frac{1}{\sqrt{2\pi}\sigma Y_i} \exp\left[-\frac{\left(\log Y_i-\mu\right)^2}{2\sigma^2} \right] \right\} \\
&= (2\pi)^{-\frac{m+n}{2}} \left(\prod_{i=1}^{n} Y_i \right)^{-1} \sigma^{-m-n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i-\mu)^2 \right] \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{m} (\log Y_i-\mu)^2 \right] \\
&= (2\pi)^{-\frac{m+n}{2}} \left(\prod_{i=1}^{n} Y_i \right)^{-1} \sigma^{-m-n} \exp\left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (X_i-\mu)^2 + \sum_{i=1}^{m} (\log Y_i-\mu)^2 \right] \right\}.
\end{aligned}$$

The log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{m+n}{2} \log(2\pi) - \sum_{i=1}^{n} \log Y_i - (m+n)\log\sigma - \frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (X_i-\mu)^2 + \sum_{i=1}^{m} (\log Y_i-\mu)^2 \right].$$

(5 marks)

UNSEEN

(b) The partial derivative of $\log L(\mu, \sigma^2)$ with respect to μ is

$$\begin{aligned}
\frac{\partial \log L(\mu, \sigma^2)}{\partial \mu} &= -\frac{1}{2\sigma^2} \left[-2\sum_{i=1}^{n} (X_i-\mu) - 2\sum_{i=1}^{m} (\log Y_i-\mu) \right] \\
&= \frac{1}{\sigma^2} \left[\sum_{i=1}^{n} X_i - n\mu + \sum_{i=1}^{m} \log Y_i - m\mu \right] \\
&= \frac{1}{\sigma^2} \left[\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} \log Y_i - (m+n)\mu \right].
\end{aligned}$$

Setting this to zero and solving for μ, we obtain the estimator

$$\widehat{\mu} = \frac{1}{m+n} \left[\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} \log Y_i \right].$$

(5 marks)

UNSEEN

(c) The partial derivative of $\log L(\mu, \sigma^2)$ with respect to σ is

$$\frac{\partial \log L(\mu, \sigma^2)}{\partial \sigma} = -\frac{m+n}{\sigma} + \frac{1}{\sigma^3} \left[\sum_{i=1}^{n} (X_i-\mu)^2 + \sum_{i=1}^{m} (\log Y_i-\mu)^2 \right].$$

Setting this to zero and solving for σ^2, we obtain the estimator

$$\widehat{\sigma^2} = \frac{1}{m+n} \left[\sum_{i=1}^{n} (X_i-\widehat{\mu})^2 + \sum_{i=1}^{m} (\log Y_i-\widehat{\mu})^2 \right].$$

(5 marks)

UNSEEN

(d) $\widehat{\mu}$ is unbiased and consistent for μ since

$$\begin{aligned}
\text{bias}\left(\widehat{\mu}\right) &= E\left(\widehat{\mu}\right) - \mu \\
&= \frac{1}{m+n} E\left[\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} \log Y_i\right] - \mu \\
&= \frac{1}{m+n} \left[\sum_{i=1}^{n} E\left(X_i\right) + \sum_{i=1}^{m} E\left(\log Y_i\right)\right] - \mu \\
&= \frac{1}{m+n} \left[\sum_{i=1}^{n} \mu + \sum_{i=1}^{m} \mu\right] - \mu \\
&= \mu - \mu \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}\left(\widehat{\mu}\right) &= \text{Var}\left(\widehat{\mu}\right) \\
&= \frac{1}{(m+n)^2} \text{Var}\left[\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} \log Y_i\right] \\
&= \frac{1}{(m+n)^2} \left[\sum_{i=1}^{n} \text{Var}\left(X_i\right) + \sum_{i=1}^{m} \left(\log Y_i\right)\right] \\
&= \frac{1}{(m+n)^2} \left[\sum_{i=1}^{n} \sigma^2 + \sum_{i=1}^{m} \sigma^2\right] \\
&= \frac{\sigma^2}{m+n}.
\end{aligned}$$

(5 marks)

UNSEEN

(e) Note that

$$\begin{aligned}
\Pr(2X < \log Y) &= \Pr\left(2N\left(\mu, \sigma^2\right) < N\left(\mu, \sigma^2\right)\right) \\
&= \Pr\left(N\left(2\mu, 4\sigma^2\right) - N\left(\mu, \sigma^2\right) < 0\right) \\
&= \Pr\left(N\left(\mu, 5\sigma^2\right) < 0\right) \\
&= \Pr\left(\frac{N\left(\mu, 5\sigma^2\right) - \mu}{\sqrt{5}\sigma} < \frac{0 - \mu}{\sqrt{5}\sigma}\right) \\
&= \Pr\left(N(0,1) < \frac{0 - \mu}{\sqrt{5}\sigma}\right) \\
&= \Phi\left(-\frac{\mu}{\sqrt{5}\sigma}\right)
\end{aligned}$$

So, the maximum likelihood estimator of $\Pr(2X < \log Y)$ is

$$\Phi\left(-\frac{\widehat{\mu}}{\sqrt{5}\widehat{\sigma}}\right).$$

(5 marks)

UNSEEN

Solutions to Question 5 (a) Suppose we wish to test $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\mu = \mu_0$; (1 mark)

(ii) the Type II error occurs if H_0 is accepted when in fact $\mu \neq \mu_0$; (1 mark)

(iii) the significance level is the probability of type I error; (1 mark)

(iv) the power function: $\Pi(\mu) = \Pr\left(\text{Reject} H_0 \mid \mu\right)$. (1 mark)

SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, \sigma^2\right)$, where σ^2 is assumed known.

(i) The rejection region for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ is

$$\frac{\sqrt{n}}{\sigma}\left|\overline{X} - \mu_0\right| > z_{\alpha/2};$$

(2 marks)

(ii) The rejection region for $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$ is

$$\frac{\sqrt{n}}{\sigma}\left(\overline{X} - \mu_0\right) < -z_{\alpha}.$$

(2 marks)

SEEN

UP TO THIS BOOK WORK

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, \sigma^2\right)$, where σ^2 is assumed known.

(i) The power function, $\Pi(\mu)$, for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ is

$$\begin{aligned}
\Pi(\mu) &= \Pr\left(\frac{\sqrt{n}}{\sigma}\left|\overline{X} - \mu_0\right| > z_{\alpha/2} \mid \mu\right) \\
&= \Pr\left(\left|\overline{X} - \mu_0\right| > \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \mid \mu\right) \\
&= \Pr\left(\overline{X} - \mu_0 > \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \mid \mu\right) + \Pr\left(\overline{X} - \mu_0 < -\frac{\sigma}{\sqrt{n}} z_{\alpha/2} \mid \mu\right) \\
&= \Pr\left(\overline{X} > \mu_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \mid \mu\right) + \Pr\left(\overline{X} < \mu_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \mid \mu\right) \\
&= \Pr\left(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{\mu_0 - \mu + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}}{\frac{\sigma}{\sqrt{n}}} \mid \mu\right) + \Pr\left(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu_0 - \mu - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}}{\frac{\sigma}{\sqrt{n}}} \mid \mu\right) \\
&= \Pr\left(N(0,1) > \frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} + z_{\alpha/2}\right) + \Pr\left(N(0,1) < \frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} - z_{\alpha/2}\right) \\
&= 1 - \Pr\left(N(0,1) < \frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} + z_{\alpha/2}\right) + \Pr\left(N(0,1) < \frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} - z_{\alpha/2}\right) \\
&= 1 - \Phi\left(\frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} + z_{\alpha/2}\right) + \Phi\left(\frac{\sqrt{n}\left(\mu_0 - \mu\right)}{\sigma} - z_{\alpha/2}\right).
\end{aligned}$$

(6 marks)

UNSEEN

(ii) The power function, $\Pi(\mu)$, for $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$ is

$$
\begin{aligned}
\Pi(\mu) &= \Pr\left(\frac{\sqrt{n}}{\sigma}\left(\overline{X}-\mu_0\right) < -z_\alpha \mid \mu\right) \\
&= \Pr\left(\left(\overline{X}-\mu_0\right) > -\frac{\sigma}{\sqrt{n}} z_\alpha \mid \mu\right) \\
&= \Pr\left(\overline{X} < \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha \mid \mu\right) \\
&= \Pr\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu_0 - \mu - \frac{\sigma}{\sqrt{n}} z_\alpha}{\frac{\sigma}{\sqrt{n}}} \mid \mu\right) \\
&= \Pr\left(N(0,1) < \frac{\sqrt{n}\left(\mu_0-\mu\right)}{\sigma} - z_\alpha\right) \\
&= \Pr\left(N(0,1) < \frac{\sqrt{n}\left(\mu_0-\mu\right)}{\sigma} - z_\alpha\right) \\
&= \Phi\left(\frac{\sqrt{n}\left(\mu_0-\mu\right)}{\sigma} - z_\alpha\right).
\end{aligned}
$$

(3 marks)

UNSEEN

(d) They are most powerful in the sense of NP lemma. The rejection rule

$$
\frac{L\left(\mu_0\right)}{L\left(\mu_1\right)} = \frac{\displaystyle\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\left(X_i-\mu_0\right)^2}{2\sigma^2}\right\}}{\displaystyle\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\left(X_i-\mu_1\right)^2}{2\sigma^2}\right\}} < k,
$$

where $\mu_1 \neq \mu_0$, along with the exact result $\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$ follows the standard normal distribution will lead to the rule in b(i). Show the full derivation.

(4 marks)

Similarly, the rejection rule

$$
\frac{L\left(\mu_0\right)}{L\left(\mu_1\right)} = \frac{\displaystyle\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\left(X_i-\mu_0\right)^2}{2\sigma^2}\right\}}{\displaystyle\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\left(X_i-\mu_1\right)^2}{2\sigma^2}\right\}} < k,
$$

where $\mu_1 < \mu_0$, along with the exact result $\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$ follows the standard normal distribution will lead to the rule in b(ii). Show the full derivation.

(4 marks)

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Solutions to Question 6

(a) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ if

$$\frac{L(\theta_1)}{L(\theta_2)} = \frac{\prod_{i=1}^{n} f(X_i; \theta_1)}{\prod_{i=1}^{n} f(X_i; \theta_2)} < k$$

for some k. (4 marks)

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UP TO THIS BOOK WORK.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $f(x) = \exp(\theta - x)$, $x > \theta > 0$, where θ is unknown.

(i) The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, $\theta_1 < \theta_2$ if

$$\begin{aligned}
\frac{L(\theta_1)}{L(\theta_2)} &= \frac{\prod_{i=1}^{n} \exp(\theta_1 - X_i) I\{X_i > \theta_1\}}{\prod_{i=1}^{n} \exp(\theta_2 - X_i) I\{X_i > \theta_2\}} \\
&= \frac{\exp(n\theta_1) \prod_{i=1}^{n} I\{X_i > \theta_1\}}{\exp(n\theta_2) \prod_{i=1}^{n} I\{X_i > \theta_2\}} \\
&= \frac{\exp(n\theta_1) I\{\min(X_1, \ldots, X_n) > \theta_1\}}{\exp(n\theta_2) I\{\min(X_1, \ldots, X_n) > \theta_2\}} \\
&= \begin{cases} \text{undefined}, & \text{if } \min(X_1, \ldots, X_n) \leq \theta_1, \\ \infty, & \text{if } \theta_1 < \min(X_1, \ldots, X_n) \leq \theta_2, \\ \exp(n\theta_1 - n\theta_2), & \text{if } \min(X_1, \ldots, X_n) > \theta_2 \end{cases} \\
&< k.
\end{aligned}$$

Draw the graph. If $\exp(n\theta_1 - n\theta_2) < k$ then the rejection region is $\min(X_1, \ldots, X_n) > \theta_2$. If $\exp(n\theta_1 - n\theta_2) \geq k$ then the rejection region is the empty set. (6 marks)

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(ii) The power function for the rejection rule in part (i) is

$$\begin{aligned}
\Pi(\theta) &= \Pr(\min(X_1, \ldots, X_n) > c \mid \theta) \\
&= \Pr(X_1 > c, \ldots, X_n > c \mid \theta) \\
&= \Pr(X_1 > c \mid \theta) \cdots \Pr(X_n > c \mid \theta) \\
&= \left[\int_c^\infty \exp(\theta - x) dx\right] \cdots \left[\int_c^\infty \exp(\theta - x) dx\right] \\
&= \exp(\theta - c) \cdots \exp(\theta - c) \\
&= \exp(n\theta - nc).
\end{aligned}$$

Note that $\Pi(\theta) = 1$ if $c < \theta$. (6 marks)

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(iii) If $n = 10$, $\theta_1 = 1$ and $\alpha = 0.05$ then

$$\exp(10 \cdot 1 - 10 \cdot c) = 0.05$$

which implies

$$10 - 10c = \log 0.05$$

which implies

$$c = 1.299573.$$

(4 marks)

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(iv) If $n = 10$, $\theta_1 = 1$, $\theta_2 = 2$ and $\alpha = 0.05$ then

$$\begin{aligned}
\Pr(\text{TypeIIerror}) &= \Pr(\text{Accept} H_0 \mid H_1 \text{istrue}) \\
&= 1 - \Pr(\text{Reject} H_0 \mid H_1 \text{istrue}) \\
&= 1 - \Pi(2) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

(5 marks)

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