SOLUTIONS TO STATISTICAL METHODS EXAM

Solutions to Question 1 Let X denote a random variable with its probability density function given by

$$f_X(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$$

for $x > 0$, $\lambda > 0$ and $\mu > 0$. X is said to have the inverse Gaussian distribution with parameters λ and μ.

(i) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= \int_0^\infty \exp(tx)\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right] dx \\
&= \int_0^\infty \exp(tx)\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda x}{2\mu^2} - \frac{\lambda}{2x} + \frac{\lambda}{\mu}\right] dx \\
&= \exp\left[\frac{\lambda}{\mu}\right] \int_0^\infty \exp(tx)\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda x}{2\mu^2} - \frac{\lambda}{2x}\right] dx \\
&\vdots \\
&\vdots \\
&\vdots \\
&= \exp\left[\frac{\lambda}{\mu} - \frac{\lambda}{\mu_t}\right] \int_0^\infty \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda x}{2\mu_t^2} - \frac{\lambda}{2x} + \frac{\lambda}{\mu_t}\right] dx \\
&= \exp\left[\frac{\lambda}{\mu} - \frac{\lambda}{\mu_t}\right] \\
&= \exp\left\{\frac{\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 t}{\lambda}}\right]\right\},
\end{aligned}$$

where $\mu_t = \sqrt{\lambda}\mu / \sqrt{\lambda - 2\mu^2 t}$.

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(ii) The first and second derivatives of $M_X(t)$ are

$$M_X^{'}(t) = \mu M_X(t)\left(1 - \frac{2t\mu^2}{\lambda}\right)^{-1/2}$$

and

$$M_X^{''}(t) = \mu^2 M_X(t)\left(1 - \frac{2t\mu^2}{\lambda}\right)^{-1} + \frac{\mu^3}{\lambda} M_X(t)\left(1 - \frac{2t\mu^2}{\lambda}\right)^{-3/2},$$

so

$$M_X^{'}(0) = \mu M_X(0)\,(1-0)^{-1/2} = \mu$$

and

$$M_X''(0) = \mu^2 M_X(0)\,(1-0)^{-1} + \frac{\mu^3}{\lambda} M_X(0)\,(1-0)^{-3/2} = \mu^2 + \frac{\mu^3}{\lambda}.$$

UNSEEN

(iii) The moment generating function of Y is

$$\begin{aligned}
M_Y(t) &= E\left[\exp(tY)\right] \\
&= E\left[\exp\left(t\left(X_1 + \cdots + X_n\right)\right)\right] \\
&= E\left[\exp\left(tX_1\right)\cdots\exp\left(tX_n\right)\right] \\
&= E\left[\exp\left(tX_1\right)\right]\cdots E\left[\exp\left(tX_n\right)\right] \\
&= M_X(t)\cdots M_X(t) \\
&= M_X^n(t) \\
&= \exp\left\{\frac{n\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 t}{\lambda}}\right]\right\} \\
&= \exp\left\{\frac{n^2\lambda}{n\mu}\left[1 - \sqrt{1 - \frac{2n^2\mu^2 t}{n^2\lambda}}\right]\right\}.
\end{aligned}$$

UNSEEN

(iv) The mean and variance of Y are

$$E\left[X_1 + \cdots + X_n\right] = E\left(X_1\right) + \cdots + E\left(X_n\right) = nE(X) = n\mu$$

and

$$\begin{aligned}
Var\left[X_1 + \cdots + X_n\right] &= Var\left(X_1\right) + \cdots + Var\left(X_n\right) \\
&= nVar(X) \\
&= n\left[M_X''(0) - \left(M_X'(0)\right)^2\right] \\
&= \frac{n\mu^3}{\lambda}.
\end{aligned}$$

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(v) Inverse Gaussian distribution with parameters $n\mu$ and $n^2\lambda$.
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Solutions to Question 2 Suppose $\widehat{\theta}$ is an estimator of θ.

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E(\widehat{\theta}) = \theta$.

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ if $\lim_{n\to\infty} E(\widehat{\theta}) = \theta$.

(iii) the bias of $\widehat{\theta}$ is $E(\widehat{\theta}) - \theta$.

(iv) the mean squared error of $\widehat{\theta}$ is $E(\widehat{\theta} - \theta)^2$.

(v) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E(\widehat{\theta} - \theta)^2 = 0$.

UP TO THIS BOOK WORK.

(b) (i) Let $Z = \min\left(|X_1|, |X_2|\right)$. Then

$$\begin{aligned} F_Z(z) &= \Pr\left[\min\left(|X_1|, |X_2|\right) < z\right] \\ &= 1 - \Pr\left[\min\left(|X_1|, |X_2|\right) > z\right] \\ &= 1 - \Pr\left[|X_1| > z, |X_2| > z\right] \\ &= 1 - \left[\Pr\left(|X| > z\right)\right]^2 \\ &= 1 - \left[1 - \Pr\left(|X| < z\right)\right]^2 \\ &= 1 - \left[1 - \frac{z}{\theta}\right]^2 \\ &= 1 - \frac{(\theta - z)^2}{\theta^2} \end{aligned}$$

and

$$f_Z(z) = \frac{2(\theta - z)}{\theta^2}$$

and

$$E(Z) = \int_0^\theta \frac{2z(\theta - z)}{\theta^2} = \frac{2}{\theta^2}\left[\frac{\theta z^2}{2} - \frac{z^3}{3}\right]_0^\theta = \frac{\theta}{3}$$

and

$$E\left(Z^2\right) = \int_0^\theta \frac{2z^2(\theta - z)}{\theta^2} dz = \frac{2}{\theta^2}\left[\frac{\theta z^3}{3} - \frac{z^4}{4}\right]_0^\theta = \frac{\theta^2}{6}.$$

So, $Bias\left(\widehat{\theta_1}\right) = 0$ and $MSE\left(\widehat{\theta_1}\right) = \theta^2/2$.

UNSEEN

(b) (ii) Let $Z = \max(X_1, X_2)$. Then

$$\begin{aligned} F_Z(z) &= \Pr\left[\max(X_1, X_2) < z\right] \\ &= \Pr\left[X_1 < z, X_2 < z\right] \\ &= \left[\Pr(X < z)\right]^2 \\ &= \left[\frac{z+\theta}{2\theta}\right]^2 \end{aligned}$$

and

$$f_Z(z) = \frac{z+\theta}{2\theta^2}$$

and

$$E(Z) = \int_{-\theta}^{\theta} z \frac{z+\theta}{2\theta^2} dz = \left[\frac{z^3}{6\theta^2} + \frac{z^2}{4\theta}\right]_{-\theta}^{\theta} = \frac{\theta}{3}$$

and

$$E\left(Z^2\right) = \int_{-\theta}^{\theta} z^2 \frac{z+\theta}{2\theta^2} dz = \left[\frac{z^4}{8\theta^2} + \frac{z^3}{6\theta}\right]_{-\theta}^{\theta} = \frac{\theta^2}{3}.$$

So, $Bias\left(\widehat{\theta}_2\right) = 0$ and $MSE\left(\widehat{\theta}_2\right) = 2\theta^2$.

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(b) (iii) Both estimators are equally good with respect to bias.

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(b) (iv) $\widehat{\theta}_1$ has smaller MSE.

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Solutions to Question 3

(a) The likelihood function is

$$L(a) = a^{-n} \left(\prod_{i=1}^{n} x_i \right)^{1/a-1}.$$

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(b) The log likelihood function is

$$\log L(a) = -n \log a + \left(a^{-1} - 1 \right) \sum_{i=1}^{n} \log x_i.$$

The derivative of $\log L$ with respect to a is

$$\frac{d \log L(a)}{da} = -\frac{n}{a} - a^{-2} \sum_{i=1}^{n} \log x_i.$$

Setting this to zero and solving for a, we obtain

$$\widehat{a} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i.$$

This is indeed an MLE since

$$\begin{aligned} \frac{d^2 \log L(a)}{da^2} &= \frac{n}{a^2} + 2a^{-3} \sum_{i=1}^{n} \log x_i \\ &= \frac{n}{\widehat{a}^2} - 2n\widehat{a}^{-2} \\ &< 0 \end{aligned}$$

at $a = \widehat{a}$.

UNSEEN

(c) The expected value is

$$
\begin{aligned}
E(\widehat{a}) &= -\frac{1}{n}\sum_{i=1}^{n} E(\log x_i) \\
&= -\frac{1}{na}\sum_{i=1}^{n}\int_0^1 \log x x^{1/a-1} dx \\
&= -\frac{1}{na}\sum_{i=1}^{n}\frac{d}{db}\int_0^1 x^b x^{1/a-1} dx\Big|_{b=0} \\
&= -\frac{1}{na}\sum_{i=1}^{n}\frac{d}{db}\int_0^1 x^{b+1/a-1} dx\Big|_{b=0} \\
&= -\frac{1}{na}\sum_{i=1}^{n}\frac{d}{db}\left[\frac{x^{b+1/a}}{b+1/a}\right]_0^1\Bigg|_{b=0} \\
&= -\frac{1}{na}\sum_{i=1}^{n}\frac{d}{db}\frac{1}{b+1/a}\Big|_{b=0} \\
&= \frac{1}{na}\sum_{i=1}^{n}\frac{1}{(b+1/a)^2}\Big|_{b=0} \\
&= \frac{1}{na}\sum_{i=1}^{n} a^2 \\
&= a.
\end{aligned}
$$

UNSEEN

(d) The variance is

$$
\begin{aligned}
Var(\widehat{a}) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(\log x_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \left\{ E\left[(\log x_i)^2\right] - a^2 \right\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n} E\left[(\log x_i)^2\right] - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \int_0^1 (\log x)^2 x^{1/a-1} dx - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \frac{d^2}{db^2} \int_0^1 x^b x^{1/a-1} dx \bigg|_{b=0} - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \frac{d^2}{db^2} \int_0^1 x^{b+1/a-1} dx \bigg|_{b=0} - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \frac{d^2}{db^2} \left[\frac{x^{b+1/a}}{b+1/a}\right]_0^1 \Bigg|_{b=0} - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \frac{d^2}{db^2} \frac{1}{b+1/a} \bigg|_{b=0} - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} \frac{2}{(b+1/a)^3} \bigg|_{b=0} - \frac{a^2}{n} \\
&= \frac{1}{n^2 a}\sum_{i=1}^{n} 2a^3 - \frac{a^2}{n} \\
&= \frac{a^2}{n}
\end{aligned}
$$

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(e) Bias $(\widehat{a}) = 0$ and MSE $(\widehat{a}) \to 0$, so the estimator is unbiased and consistent.

UNSEEN

(f) Note that

$$
\Pr(X < 0.5) = \int_0^{0.5} a^{-1} x^{a^{-1}-1} dx = \left[x^{a^{-1}}\right]_0^{0.5} = 0.5^{1/a},
$$

which is a one-to-one function of a for $a > 0$. By the invariance principle, its maximum likelihood estimator is $0.5^{n/\left(\sum_{i=1}^{n} \log x_i\right)}$.

UNSEEN

Solutions to Question 4

(a) The joint likelihood function is

$$\begin{aligned} L\left(\mu,\sigma^2\right) &= \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{\left(X_i-\mu\right)^2}{2\sigma^2}\right\}\right]\prod_{i=1}^{n}\left[\frac{1}{\sigma^2}\exp\left\{-\frac{Y_i}{\sigma^2}\right\}\right] \\ &= \frac{1}{(2\pi)^{n/2}\sigma^{3n}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2-\frac{1}{\sigma^2}\sum_{i=1}^{n}Y_i\right\}. \end{aligned}$$

UNSEEN

(b) The log likelihood function is

$$\log L = -\frac{n}{2}\log(2\pi) - 3n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{n}Y_i.$$

The partial derivative of $\log L$ with respect to σ is

$$\frac{\partial\log L}{\partial\sigma} = -\frac{3n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 + \frac{2}{\sigma^3}\sum_{i=1}^{n}Y_i.$$

Setting this to zero and solving for σ, we obtain

$$\widehat{\sigma^2} = \frac{1}{3n}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 + \frac{2}{3n}\sum_{i=1}^{n}Y_i.$$

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(c) The partial derivative of $\log L$ with respect to μ is

$$\frac{\partial\log L}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right).$$

Setting this to zero and solving for μ, we obtain

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}X_i = \overline{X}.$$

UNSEEN

(d) Note that

$$\begin{aligned} \widehat{\sigma^2} &= \frac{1}{3n}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 + \frac{2}{3n}\sum_{i=1}^{n}Y_i \\ &= \frac{\sigma^2}{3n}\frac{(n-1)S^2}{\sigma^2} + \frac{2}{3n}\sum_{i=1}^{n}Y_i \\ &= \frac{\sigma^2}{3n}\chi^2_{n-1} + \frac{2}{3n}\sum_{i=1}^{n}Y_i, \end{aligned}$$

so

$$
\begin{aligned}
E\left[\widehat{\sigma^2}\right] &= \frac{\sigma^2}{3n}E\left[\chi^2_{n-1}\right] + \frac{2}{3n}E\left[\sum_{i=1}^{n} Y_i\right] \\
&= \frac{\sigma^2}{3n}E\left[\chi^2_{n-1}\right] + \frac{2}{3n}\sum_{i=1}^{n} E\left[Y_i\right] \\
&= \frac{\sigma^2}{3n}(n-1) + \frac{2}{3n}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{(n-1)\sigma^2}{3n} + \frac{2\sigma^2}{3} \\
&= \sigma^2 - \frac{\sigma^2}{3n}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[\widehat{\sigma^2}\right] &= \frac{\sigma^4}{9n^2}Var\left[\chi^2_{n-1}\right] + \frac{4}{9n^2}Var\left[\sum_{i=1}^{n} Y_i\right] \\
&= \frac{2\sigma^4(n-1)}{9n^2} + \frac{4}{9n^2}\sum_{i=1}^{n} Var\left[Y_i\right] \\
&= \frac{2\sigma^4(n-1)}{9n^2} + \frac{4}{9n^2}\sum_{i=1}^{n} \sigma^4 \\
&= \frac{2\sigma^4(n-1)}{9n^2} + \frac{4\sigma^4}{9n}
\end{aligned}
$$

and

$$
MSE\left[\widehat{\sigma^2}\right] = \frac{2\sigma^4(n-1)}{9n^2} + \frac{4\sigma^4}{9n} + \frac{\sigma^4}{9n^2}
$$

Hence, $\widehat{\sigma^2}$ is biased and but consistent.

UNSEEN

(e) Note that

$$
E\left[\overline{X}\right] = \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu
$$

and

$$
Var\left[\overline{X}\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var\left[X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}
$$

Hence, $\overline{X}$ is unbiased and consistent.

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Solutions to Question 5 (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\theta = \theta_0$.

(ii) the Type II error occurs if H_0 is accepted when in fact $\theta \neq \theta_0$.

(iii) the significance level is the probability of type I error.

(iv) the power function: $\Pi(\theta) = \Pr(\text{ Reject } H_0 \mid \theta)$.

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(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with parameter p. Assume $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ has a normal distribution with mean p and variance $p(1-p)/n$.

(i) The rejection region for $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,|\bar{x} - p_0| > z_{\alpha/2}.$$

(ii) The rejection region for $H_0 : p = p_0$ versus $H_1 : p < p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x} - p_0) < -z_{\alpha}.$$

(iii) The rejection region for $H_0 : p = p_0$ versus $H_1 : p > p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x} - p_0) > z_{\alpha}.$$

UP TO THIS BOOK WORK.

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(c) (i) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,|\bar{x}-p_0| > z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(|\bar{x}-p_0| > \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(\bar{x} > p_0 + \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2} \text{ or } \bar{x} < p_0 - \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right. \\
&\qquad \left.\text{or } \sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(Z > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right. \\
&\qquad \left.\text{or } Z < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\,\middle|\, p\right) \\
&= 1 - \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right) \\
&\qquad + \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

UNSEEN

(c) (ii) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p < p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x}-p_0) < -z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(\bar{x} < p_0 - \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(Z < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\,\middle|\, p\right) \\
&= \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

UNSEEN

(c) (iii) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p > p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x} - p_0) > z_\alpha \,\middle|\, p\right) \\
&= \Pr\left(\bar{x} > p_0 + \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_\alpha \,\middle|\, p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x} - p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0 - p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \,\middle|\, p\right) \\
&= \Pr\left(Z > \sqrt{n}\frac{p_0 - p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \,\middle|\, p\right) \\
&= 1 - \Phi\left(\sqrt{n}\frac{p_0 - p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

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They are not most powerful. Since we have used the fact $\sqrt{n}\{\bar{x} - p\}/\sqrt{p(1-p)}$ can be approximate by the the standard normal distribution. The exact distribution of $\bar{x}$ is not normal.

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Solutions to Question 6 The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ in favor of $H_1 : \theta = \theta_2$ if

$$\frac{L(\theta_1)}{L(\theta_2)} = \frac{\displaystyle\prod_{i=1}^{n} f(X_i; \theta_1)}{\displaystyle\prod_{i=1}^{n} f(X_i; \theta_2)} < k$$

for some k. UP TO THIS BOOK WORK.

(b) (i) Note that

$$\begin{aligned}
\frac{L(K)}{L(L)} &= \frac{\frac{aK^a}{X_1^{a+1}} I\{X_1 > K\} \cdots \frac{aK^a}{X_n^{a+1}} I\{X_n > K\}}{\frac{aL^a}{X_1^{a+1}} I\{X_1 > L\} \cdots \frac{aL^a}{X_n^{a+1}} I\{X_n > L\}} \\
&= \frac{K^{na} I\{X_1 > K\} \cdots I\{X_n > K\}}{L^{na} I\{X_1 > L\} \cdots I\{X_n > L\}} \\
&= \frac{K^{na} I\{\min(X_1, \ldots, X_n) > K\}}{L^{na} I\{\min(X_1, \ldots, X_n) > L\}}.
\end{aligned}$$

This is an increasing function of $\min(X_1, \ldots, X_n)$. So, by the N-P lemma, we reject H_0 if

$$\frac{K^{na} I\{\min(X_1, \ldots, X_n) > K\}}{L^{na} I\{\min(X_1, \ldots, X_n) > L\}} < k$$

if and only if

$$\min(X_1, \ldots, X_n) < c$$

for some c.

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(b) (ii) The power function is

$$\begin{aligned}
\Pi(\theta) &= \Pr[\min(X_1, \ldots, X_n) < c \mid \theta] \\
&= 1 - \Pr[\min(X_1, \ldots, X_n) > c \mid \theta] \\
&= 1 - \Pr[X_1 > c, \ldots, X_n > c \mid \theta] \\
&= 1 - (\Pr[X > c \mid \theta])^n \\
&= 1 - \left(\int_c^{\infty} \frac{a\theta^a}{x^{a+1}} dx \right)^n \\
&= 1 - \left(\left[-\frac{\theta^a}{x^a} \right]_c^{\infty} \right)^n \\
&= 1 - \left(\frac{\theta^a}{c^a} \right)^n \\
&= 1 - \frac{\theta^{na}}{c^{na}}.
\end{aligned}$$

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(b) (iii) If $n = 10$, $a = 1$, $K = 1$ and $\alpha = 0.05$ then

$$1 - \frac{1}{c^{10}} = 0.05$$

which implies

$$\frac{1}{c^{10}} = 0.95$$

which implies

$$c^{10} = 1/0.95$$

which implies $c = (1/0.95)^{1/10} = 1.005143$.

UNSEEN

(b) (iv) If $n = 10$, $a = 1$, $K = 1$, $L = 1/2$ and $\alpha = 0.05$ then

$$\begin{aligned}
\Pr(Type II error) &= \Pr(\text{Accept } H_0 \mid \theta = 1/2) \\
&= 1 - \Pr(\text{Reject } H_0 \mid \theta = 1/2) \\
&= 1 - \Pi(1/2) \\
&= 1 - \left[1 - \frac{(1/2)^{10}}{c^{10}}\right] \\
&= 0.95 \cdot 2^{-10} \\
&= 0.0009277344.
\end{aligned}$$

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