SOLUTIONS TO
STATISTICAL METHODS EXAM

Solutions to Question 1

(i) Setting $z = \exp\{-(y-\mu)/\beta\}$, we obtain the cumulative distribution function as

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} \frac{1}{\beta} \exp\left(-\frac{y-\mu}{\beta}\right) \exp\left\{-\exp\left(-\frac{y-\mu}{\beta}\right)\right\} dy \\ &= \int_{\exp\{-(x-\mu)/\beta\}}^{\infty} \frac{1}{\beta} z \exp\{-z\} \frac{\beta}{z} dz \\ &= \int_{\exp\{-(x-\mu)/\beta\}}^{\infty} \exp\{-z\} dz \\ &= \left[-\exp(-z)\right]_{\exp\{-(x-\mu)/\beta\}}^{\infty} \\ &= \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\}. \end{aligned}$$

(ii) Setting $z = \exp\{-(x-\mu)/\beta\}$, we obtain the moment generating function as

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} \frac{1}{\beta} \exp\left(tx - \frac{x-\mu}{\beta}\right) \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\} dx \\ &= \int_{0}^{\infty} \frac{1}{\beta} \exp\{t(\mu - \beta \log z)\} z \exp\{-z\} \frac{\beta}{z} dz \\ &= \exp(\mu t) \int_0^{\infty} \exp\{-t\beta \log z\} \exp\{-z\} dz \\ &= \exp(\mu t) \int_0^{\infty} z^{-\beta t} \exp\{-z\} dz \\ &= \exp(\mu t) \Gamma(1-\beta t), \end{aligned}$$

where the last step follows by the definition of the gamma function.

(iii) the first derivative of $M_X(t)$ is

$$M_X^{'}(t) = -\beta \exp(\mu t) \Gamma^{'}(1-\beta t) + \mu \exp(\mu t) \Gamma(1-\beta t).$$

So, $E(X) = M_X^{'}(0) = \mu - \beta \Gamma^{'}(1)$.

(iv) Let $Z = \max(X_1, X_2, \ldots, X_n)$. The cdf of Z is

$$\begin{aligned} F_Z(z) &= \Pr\left[\max(X_1, X_2, \ldots, X_n) \le z\right] \\ &= \Pr\left[X_1 \le z, X_2 \le z, \ldots, X_n \le z\right] \\ &= \Pr\left[X_1 \le z\right] \Pr\left[X_2 \le z\right] \cdots \Pr\left[X_n \le z\right] \\ &= F_X(z) F_X(z) \cdots F_X(z) \\ &= F_X^n(z) \\ &= \exp\left\{-n \exp\left(-\frac{z-\mu}{\beta}\right)\right\} \\ &= \exp\left\{-\exp\left(-\frac{z-(\mu+\beta\log n)}{\beta}\right)\right\}, \end{aligned}$$

so the result follows.

Solutions to Question 2 Suppose $\widehat{\theta}$ is an estimator of θ.

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E(\widehat{\theta}) = \theta$.

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ if $\lim_{n\to\infty} E(\widehat{\theta}) = \theta$.

(iii) the bias of $\widehat{\theta}$ is $E(\widehat{\theta}) - \theta$.

(iv) the mean squared error of $\widehat{\theta}$ is $E(\widehat{\theta} - \theta)^2$.

(v) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E(\widehat{\theta} - \theta)^2 = 0$.

UP TO THIS BOOK WORK.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Exp (λ) distribution. Consider the following estimators for $\theta = 1/\lambda$: $\widehat{\theta_1} = (1/n)\sum_{i=1}^n X_i$ and $\widehat{\theta_2} = (1/(n+1))\sum_{i=1}^n X_i$.

(i) The bias of $\widehat{\theta_1}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_1}\right) - \theta &= E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) - \theta \\
&= \frac{1}{n}\sum_{i=1}^n E\left(X_i\right) - \theta \\
&= \frac{1}{n}\sum_{i=1}^n \theta - \theta \\
&= \theta - \theta \\
&= 0.
\end{aligned}
$$

The bias of $\widehat{\theta_2}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_2}\right) - \theta &= E\left(\frac{1}{n+1}\sum_{i=1}^n X_i\right) - \theta \\
&= \frac{1}{n+1}\sum_{i=1}^n E\left(X_i\right) - \theta \\
&= \frac{1}{n+1}\sum_{i=1}^n \theta - \theta \\
&= \frac{n\theta}{n+1} - \theta \\
&= -\frac{\theta}{n+1}.
\end{aligned}
$$

(ii) The variance of $\widehat{\theta}_1$ is

$$\begin{aligned}
Var\left(\widehat{\theta}_1\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \theta^2 \\
&= \frac{\theta^2}{n}.
\end{aligned}$$

The variance of $\widehat{\theta}_2$ is

$$\begin{aligned}
Var\left(\widehat{\theta}_2\right) &= Var\left(\frac{1}{n+1}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{(n+1)^2}\sum_{i=1}^{n} Var\left(X_i\right) \\
&= \frac{1}{(n+1)^2}\sum_{i=1}^{n} \theta^2 \\
&= \frac{n\theta^2}{(n+1)^2}.
\end{aligned}$$

(iii) The mean squared error of $\widehat{\theta}_1$ is

$$MSE\left(\widehat{\theta}_1\right) = \frac{\theta^2}{n}.$$

The mean squared error of $\widehat{\theta}_2$ is

$$MSE\left(\widehat{\theta}_2\right) = \frac{n\theta^2}{(n+1)^2} + \left(\frac{\theta}{n+1}\right)^2 = \frac{\theta^2}{n+1}.$$

(iv) In terms of bias, $\widehat{\theta}_1$ is unbiased and $\widehat{\theta}_2$ is biased (however, $\widehat{\theta}_2$ is asymptotically unbiased). So, one would prefer $\widehat{\theta}_1$ if bias is the important issue.

In terms of mean squared error, $\widehat{\theta}_2$ has better efficiency (however, both estimators are consistent). So, one would prefer $\widehat{\theta}_2$ if efficiency is the important issue.

Solutions to Question 3 Consider the two independent random samples: $X_1, X_2, \ldots, X_n$ from $N(\mu_X, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_m$ from $N(\mu_Y, \sigma^2)$, where σ^2 is assumed known. The parameters μ_X and μ_Y are assumed not known.

(i) The likelihood function of μ_X and μ_Y is

$$
\begin{aligned}
L(\mu_X, \mu_Y) &= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(X_i - \mu_X)^2}{2\sigma^2} \right\} \right) \left(\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(Y_i - \mu_Y)^2}{2\sigma^2} \right\} \right) \\
&= \frac{1}{(2\pi)^{(m+n)/2}\sigma^{m+n}} \exp\left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (X_i - \mu_X)^2 + \sum_{i=1}^{m} (Y_i - \mu_Y)^2 \right] \right\}.
\end{aligned}
$$

(ii) The log likelihood function of μ_X and μ_Y is

$$
l(\mu_X, \mu_Y) = -\frac{m+n}{2} \log(2\pi) - (m+n) \log \sigma - \frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (X_i - \mu_X)^2 + \sum_{i=1}^{m} (Y_i - \mu_Y)^2 \right].
$$

The partial derivatives with respect to μ_X and μ_Y are

$$
\frac{\partial l(\mu_X, \mu_Y)}{\partial \mu_X} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu_X)
$$

and

$$
\frac{\partial l(\mu_X, \mu_Y)}{\partial \mu_Y} = \frac{1}{\sigma^2} \sum_{i=1}^{m} (Y_i - \mu_Y).
$$

Setting $\partial l(\mu_X, \mu_Y)/\partial \mu_X = 0$, one obtains

$$
\begin{aligned}
& \sum_{i=1}^{n} (X_i - \mu_X) = 0 \\
\Rightarrow\ & n\mu_X = \sum_{i=1}^{n} X_i \\
\Rightarrow\ & \mu_X = \bar{X}
\end{aligned}
$$

where $\bar{X} = (1/n) \sum_{i=1}^{n} X_i$. Similarly, setting $\partial l(\mu_X, \mu_Y)/\partial \mu_Y = 0$, one obtains

$$
\begin{aligned}
& \sum_{i=1}^{m} (Y_i - \mu_Y) = 0 \\
\Rightarrow\ & m\mu_Y = \sum_{i=1}^{m} Y_i \\
\Rightarrow\ & \mu_Y = \bar{Y}
\end{aligned}
$$

where $\bar{Y} = (1/m) \sum_{i=1}^{m} Y_i$. So, the mles are $\widehat{\mu_X} = \bar{X}$ and $\widehat{\mu_Y} = \bar{Y}$.

(iii) Note $X - Y \sim N(\mu_X - \mu_Y, 2\sigma^2)$. So,

$$\begin{aligned}
\Pr(X < Y) &= \Pr(X - Y < 0) \\
&= \Pr\left(\frac{X - Y - (\mu_X - \mu_Y)}{\sqrt{2}\sigma} < \frac{0 - (\mu_X - \mu_Y)}{\sqrt{2}\sigma}\right) \\
&= \Pr\left(Z < \frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right) \\
&= \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right)
\end{aligned}$$

and so the mle of $\Pr(X < Y)$ is $\Phi((\widehat{\mu_Y} - \widehat{\mu_X})/(\sqrt{2}\sigma))$.

(iv) Note $\bar{X} \sim N(\mu_X, \sigma^2/n)$ and so $E(\widehat{\mu_X}) = \mu_X$ and $Var(\widehat{\mu_X}) = \sigma^2/n$. So, $\widehat{\mu_X}$ is an unbiased and consistent estimator for μ_X.

(v) Note $\bar{Y} \sim N(\mu_Y, \sigma^2/m)$ and so $E(\widehat{\mu_Y}) = \mu_Y$ and $Var(\widehat{\mu_Y}) = \sigma^2/m$. So, $\widehat{\mu_Y}$ is an unbiased and consistent estimator for μ_Y.

Solutions to Question 4 Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

(i) The joint likelihood function of μ and σ^2 is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(X_i-\mu\right)^2}{2\sigma^2} \right] \right\} \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(X_i-\mu\right)^2 \right]. \end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right)^2.$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i-\mu\right) = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} X_i - n\mu\right) \tag{1}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}\left(X_i-\mu\right)^2, \tag{2}$$

respectively.

(ii) Using equation (1), one can see that the solution of $\partial \log L / \partial \mu = 0$ is $\mu = \bar{X} = (1/n)\sum_{i=1}^{n} X_i$.

(iii) Using equation (2), one can see that the solution of $\partial \log L / \partial \sigma = 0$ is $\sigma^2 = (1/n)\sum_{i=1}^{n}(X_i - \bar{X})^2$.

(iv) The mle, $\widehat{\mu}$, is an unbiased and consistent estimator for μ since

$$\begin{aligned} E\left(\widehat{\mu}\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned}
Var\left(\widehat{\mu}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{\sigma^2}{n}.
\end{aligned}$$

(v) The mle, $\widehat{\sigma^2}$, is a biased and consistent estimator for σ^2 since

$$\begin{aligned}
E\left(\widehat{\sigma^2}\right) &= E\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right] \\
&= E\left[\frac{n-1}{n}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\frac{n-1}{\sigma^2}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\chi^2_{n-1}\right] \\
&= \frac{(n-1)\sigma^2}{n}
\end{aligned}$$

and

$$\begin{aligned}
Var\left(\widehat{\sigma^2}\right) &= Var\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right] \\
&= Var\left[\frac{n-1}{n}S^2\right] \\
&= \frac{\sigma^4}{n^2}E\left[\frac{n-1}{\sigma^2}S^2\right] \\
&= \frac{\sigma^4}{n^2}E\left[\chi^2_{n-1}\right] \\
&= \frac{2(n-1)\sigma^4}{n^2}.
\end{aligned}$$

Note that we have used the fact $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Furthermore, $S^2 = (1/(n-1))\sum_{i=1}^{n}(X_i - \bar{X})^2$ denotes the sample variance.

Solutions to Question 5 (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\theta = \theta_0$.

(ii) the Type II error occurs if H_0 is accepted when in fact $\theta \neq \theta_0$.

(iii) the significance level is the probability of type I error.

(iv) the power function: $\Pi(\theta) = \Pr(\text{ Reject } H_0 \mid \theta)$.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with parameter p. Assume $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ has a normal distribution with mean p and variance $p(1-p)/n$.

(i) The rejection region for $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,|\bar{x} - p_0| > z_{\alpha/2}.$$

(ii) The rejection region for $H_0 : p = p_0$ versus $H_1 : p < p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x} - p_0) < -z_{\alpha}.$$

(iii) The rejection region for $H_0 : p = p_0$ versus $H_1 : p > p_0$ is

$$\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x} - p_0) > z_{\alpha}.$$

UP TO THIS BOOK WORK.

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with parameter p. Assume $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ has a normal distribution with mean p and variance $p(1-p)/n$.

(i) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,|\bar{x}-p_0| > z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(|\bar{x}-p_0| > \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(\bar{x} > p_0 + \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2} \text{ or } \bar{x} < p_0 - \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right. \\
&\qquad \left.\text{or } \sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\,\middle|\, p\right) \\
&= \Pr\left(Z > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right. \\
&\qquad \left.\text{or } Z < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\,\middle|\, p\right) \\
&= 1 - \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right) \\
&\qquad +\Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha/2}\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

(ii) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p < p_0$ is

$$\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x}-p_0) < -z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(\bar{x} < p_0 - \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\,\middle|\, p\right) \\
&= \Pr\left(Z < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\,\middle|\, p\right) \\
&= \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} - \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_{\alpha}\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

(iii) The power function, $\Pi(p)$, for $H_0 : p = p_0$ versus $H_1 : p > p_0$ is

$$
\begin{aligned}
\Pi(p) &= \Pr\left(\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,(\bar{x}-p_0) > z_\alpha \,\middle|\, p \right) \\
&= \Pr\left(\bar{x} > p_0 + \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} z_\alpha \,\middle|\, p \right) \\
&= \Pr\left(\sqrt{n}\frac{\bar{x}-p}{\sqrt{p(1-p)}} < \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \,\middle|\, p \right) \\
&= \Pr\left(Z > \sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \,\middle|\, p \right) \\
&= 1 - \Phi\left(\sqrt{n}\frac{p_0-p}{\sqrt{p(1-p)}} + \sqrt{\frac{\bar{x}(1-\bar{x})}{p(1-p)}} z_\alpha \right),
\end{aligned}
$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Note that we have used the fact $\sqrt{n}\{\bar{x}-p\}/\sqrt{p(1-p)}$ has the standard normal distribution.

Solutions to Question 6 The Neyman-Pearson test rejects $H_0 : \theta = \theta_1$ in favor of $H_1 : \theta = \theta_2$ if

$$\frac{L(\theta_1)}{L(\theta_2)} = \frac{\prod_{i=1}^{n} f(X_i; \theta_1)}{\prod_{i=1}^{n} f(X_i; \theta_2)} < k$$

for some k. UP TO THIS BOOK WORK.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform $(0, \theta)$ distribution.

(i) The most powerful test is to reject $H_0 : \theta = \theta_1$ if

$$\begin{aligned} \frac{L(\theta_1)}{L(\theta_2)} &= \frac{\theta_1^{-n} I\{0 < X_1 < \theta_1\} I\{0 < X_2 < \theta_1\} \cdots I\{0 < X_n < \theta_1\}}{\theta_2^{-n} I\{0 < X_1 < \theta_2\} I\{0 < X_2 < \theta_2\} \cdots I\{0 < X_n < \theta_2\}} \\ &= \frac{\theta_2^n\, I\{\max(X_1, X_2, \ldots, X_n) < \theta_1\}}{\theta_1^n\, I\{\max(X_1, X_2, \ldots, X_n) < \theta_2\}} \\ &< k_0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \frac{I\{\max(X_1, X_2, \ldots, X_n) < \theta_1\}}{I\{\max(X_1, X_2, \ldots, X_n) < \theta_2\}} < \frac{k_0 \theta_1^n}{\theta_2^n} \\ \iff & \max(X_1, X_2, \ldots, X_n) > k \end{aligned}$$

as required. The last step follows because

$$\frac{I\{\max(X_1, X_2, \ldots, X_n) < \theta_1\}}{I\{\max(X_1, X_2, \ldots, X_n) < \theta_2\}}$$

is a decreasing function of $\max(X_1, X_2, \ldots, X_n)$.

(ii) The power function is

$$\begin{aligned} \Pi(\theta) &= \Pr(\text{Reject } H_0 \mid \theta) \\ &= \Pr(\max(X_1, X_2, \ldots, X_n) > k \mid \theta) \\ &= 1 - \Pr(\max(X_1, X_2, \ldots, X_n) \leq k \mid \theta) \\ &= 1 - \Pr(X_1 \leq k \mid \theta) \Pr(X_2 \leq k \mid \theta) \cdots \Pr(X_n \leq k \mid \theta) \\ &= 1 - \left(\frac{k}{\theta}\right)^n. \end{aligned}$$

(iii) Note that

$$\begin{aligned} & 1 - (2k)^5 = 0.05 \\ \iff & 2k = (0.95)^{1/5} \\ \iff & k = (1/2)(0.95)^{1/5}. \end{aligned}$$

So, $k = 0.4948969$.

(iv) Note that

$$
\begin{aligned}
\beta &= \Pr(\text{Type II error}) \\
&= 1 - \left[1 - \left(\frac{0.4948969}{0.6} \right)^5 \right] \\
&= \left(\frac{0.4948969}{0.6} \right)^5 \\
&= 0.3817837.
\end{aligned}
$$