MATH20802

Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

1 June 2018

14:00-16:00

Answer FOUR of the SIX questions. If more than FOUR questions are attempted, then credit will be given for the best FOUR answers.

University-approved calculators may be used

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1. A random variable X is said to have the hyperbolic secant distribution if its probability density function is given by

$$f_X(x) = \frac{1}{\exp\left(\frac{\pi x}{2}\right) + \exp\left(-\frac{\pi x}{2}\right)}$$

for $-\infty < x < +\infty$.

(a) Derive the cumulative distribution function of X. (6 marks)

(b) Show that the moment generating function of X can be expressed as

$$M_X(t) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + \frac{t}{\pi}\right)\Gamma\left(\frac{1}{2} - \frac{t}{\pi}\right)$$

for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, where $\Gamma(a)$ denotes the gamma function defined by

$$\Gamma\left(a\right) = \int_0^\infty t^{a-1}\exp(-t)dt$$

for $a > 0$. (6 marks)

(c) Use your result in (b) to derive the first two moments of X. Express the moments in terms of the gamma function and its derivatives. (6 marks)

(d) A random variable Y, whose distribution is symmetric around zero, satisfies $\mid Y \mid = \exp\left(\pi X/2\right)$. Derive the probability density and cumulative distribution functions of Y. (5 marks)

(e) What is the distribution of Y known as? (2 marks)

[Total: 25 marks]

2. (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ; (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as $\text{bias}(\widehat{\theta})$); (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as $\text{MSE}(\widehat{\theta})$); (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ. (2 marks)

(b) Suppose X_1 and X_2 are independent $\text{Exp}(1/\theta)$ and Uniform $[0, \theta]$ random variables. Let $\widehat{\theta} = aX_1 + bX_2$ denote a class of estimators of θ, where a and b are constants.

(i) Determine the bias of $\widehat{\theta}$; (3 marks)

(ii) Determine the variance of $\widehat{\theta}$; (3 marks)

(iii) Determine the mean squared error of $\widehat{\theta}$; (2 marks)

(iv) Determine the condition involving a and b such that $\widehat{\theta}$ is unbiased for θ; (2 marks)

(v) Determine the value of a such that $\widehat{\theta}$ is unbiased for θ and has the smallest variance. (5 marks)

[Total: 25 marks]

3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

for $x > 0$.

(a) Write down the likelihood function of σ^2. (4 marks)

(b) Show that the maximum likelihood estimator of σ^2 is

$$\widehat{\sigma^2} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2.$$

(4 marks)

(c) Deduce the maximum likelihood estimator of σ. (1 marks)

(d) Show that the estimator in part (b) is an unbiased estimator of σ^2. (8 marks)

(e) Show that the estimator in part (b) is a consistent estimator of σ^2. (8 marks)

[Total: 25 marks]

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f_X(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$$

for $x > 0$, $\mu > 0$ and $\lambda > 0$. Assume both μ and λ are unknown.

(a) Write down the joint likelihood function of μ and λ. (5 marks)

(b) Show that the maximum likelihood estimator of μ is $\widehat{\mu} = \overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. (5 marks)

(c) Show that the maximum likelihood estimator of λ is $\widehat{\lambda} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i} - \frac{1}{\overline{X}}\right)^{-1}$. (5 marks)

(d) Show that the estimator in (b) is unbiased and consistent for μ. (5 marks)

(e) Show that the estimator $1/\widehat{\lambda}$ is biased but consistent for $1/\lambda$. You may use the following fact without proof:

$$\lambda\left(\sum_{i=1}^{n}\frac{1}{X_i} - \frac{n}{\overline{X}}\right) \sim \chi^2_{n-1}.$$

(5 marks)

[Total: 25 marks]

5. (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of a test. (2 marks)

(ii) the Type II error of a test. (2 marks)

(iii) the significance level of a test. (2 marks)

(iv) the power function of a test (denoted $\Pi(\theta)$). (2 marks)

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where σ is not known. State the rejection region for each of the following tests:

(i) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$. (2 marks)

(ii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma < \sigma_0$. (2 marks)

(iii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$. (2 marks)

In each case, assume a significance level of α.

(c) Under the same assumptions as in part (b), find the power function, $\Pi(\sigma)$, for each of the following tests:

(i) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$. (5 marks)

(ii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma < \sigma_0$. (3 marks)

(iii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$. (3 marks)

In each case, assume a significance level of α.

[Total: 25 marks]

6. (a) Describe the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x; \theta)$. State both the test statistic and the form of the rejection region. (4 marks)

(b) Suppose $X_1, X_2, \ldots, X_n$ is an independent random sample from Exp (θ), where θ is unknown.

(i) Derive the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, where $\theta_2 > \theta_1$. Show that the rule for rejecting H_0 can be expressed as $\sum_{i=1}^{n} X_i < c$. (8 marks)

(ii) Show that the power function for the rejection rule in part (i) is

$$\Pi(\theta) = 1 - \left[1 + \theta c + \frac{\theta^2 c^2}{2} + \cdots + \frac{\theta^{n-1} c^{n-1}}{(n-1)!}\right] \exp(-\theta c).$$

You may use mathematical induction to prove this.

(8 marks)

(iii) Show that if $n = 2$, $\theta_1 = 1$ and $c = 0.3553595$, then the probability of a type I error is 0.05. (2 marks)

(iv) Find the probability of a type II error if $n = 2$, $\theta_1 = 1$, $\theta_2 = 2$ and the probability of a type I error is 0.05. (3 marks)

[Total: 25 marks]

END OF EXAMINATION PAPER