**Two hours**

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

**THE UNIVERSITY OF MANCHESTER**

STATISTICAL METHODS

21 May 2015

09:45-11:45

Answer FOUR of the SIX questions. If more than FOUR questions are attempted, then credit will be given for the best FOUR answers.

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University-approved calculators may be used

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**1.** Let $X$ denote a random variable with its probability density function given by

$$f_X(x;\lambda,\mu) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$$

for $x > 0$, $\lambda > 0$ and $\mu > 0$. $X$ is said to have the inverse Gaussian distribution with parameters $\lambda$ and $\mu$.

(a) Show that

$$\exp(tx) f_X(x;\lambda,\mu) = \exp\left\{\frac{\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 t}{\lambda}}\right]\right\} f_X\left(x;\lambda,\mu_t\right),$$

where $\mu_t = \sqrt{\lambda}\mu/\sqrt{\lambda - 2\mu^2 t}$. Hence, deduce that the moment generating function of $X$ can be expressed as

$$M_X(t) = E\left[\exp(tX)\right] = \exp\left\{\frac{\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 t}{\lambda}}\right]\right\}$$

for $t < \lambda/\left(2\mu^2\right)$. (10 marks)

(b) Use your result in (a) to derive the first two moments of $X$. (4 marks)

(c) If $X_i$ are independent and identical random variables and are distributed as $X$ derive the moment generating function of $Y = X_1 + \cdots + X_n$. (4 marks)

(d) Calculate the mean and variance of $Y$. (4 marks)

(e) What is the distribution of $Y$? (3 marks)

[Total: 25 marks]

**2.** (a) Suppose $\widehat{\theta}$ is an estimator of $\theta$ based on a random sample of size $n$. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of $\theta$; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of $\theta$; (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as $\text{bias}(\widehat{\theta})$); (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as $\text{MSE}(\widehat{\theta})$); (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of $\theta$. (2 marks)

(b) Suppose $X_1$ and $X_2$ are independent Uniform$[-\theta, \theta]$ random variables. Let $\widehat{\theta_1} = 3 \min \left( \left| X_1 \right|, \left| X_2 \right| \right)$ and $\widehat{\theta_2} = 3 \max \left( X_1, X_2 \right)$ denote possible estimators of $\theta$.

(i) Derive the bias and mean squared error of $\widehat{\theta_1}$; (7 marks)

(ii) Derive the bias and mean squared error of $\widehat{\theta_2}$; (6 marks)

(iii) Which of the estimators ($\widehat{\theta_1}$ and $\widehat{\theta_2}$) is better with respect to bias and why? (1 marks)

(iv) Which of the estimators ($\widehat{\theta_1}$ and $\widehat{\theta_2}$) is better with respect to mean squared error and why? (1 marks)

[Total: 25 marks]

**3.** Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $f(x) = a^{-1}x^{a^{-1}-1}$, $0 < x < 1$, where $a > 0$ is an unknown parameter.

(a) Write down the likelihood function of $a$. (2 marks)

(b) Show that the maximum likelihood estimator of $a$ is $\widehat{a} = -\frac{1}{n}\sum_{i=1}^{n} \log X_i$. (7 marks)

(c) Derive the expected value of $\widehat{a}$ in part (b). You may use the fact that $\int_0^1 x^\alpha \log x dx = -(\alpha+1)^{-2}$ without proof. (6 marks)

(d) Derive the variance of $\widehat{a}$ in part (b). You may use the fact that $\int_0^1 x^\alpha (\log x)^2 dx = 2(\alpha+1)^{-3}$ without proof. (6 marks)

(e) Show that $\widehat{a}$ is an unbiased and a consistent estimator for $a$. (2 marks)

(f) Find the maximum likelihood estimator of $\Pr(X < 0.5)$, where $X$ has the probability density function $f(x) = a^{-1}x^{a^{-1}-1}$, $0 < x < 1$. Justify your answer. (2 marks)

[Total: 25 marks]

**4.** Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$. Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from $Exp(\sigma^{-2})$ independent of $X_1, X_2, \ldots, X_n$. Assume both $\mu$ and $\sigma^2$ are unknown.

(a) Write down the joint likelihood function of $\mu$ and $\sigma^2$. (5 marks)

(b) Show that the maximum likelihood estimator of $\mu$ is $\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$. (4 marks)

(c) Show that the maximum likelihood estimator of $\sigma^2$ is $\widehat{\sigma^2} = \frac{1}{3n}\sum_{i=1}^{n}(X_i - \widehat{\mu})^2 + \frac{2}{3n}\sum_{i=1}^{n} Y_i$. (4 marks)

(d) Show that the estimator in (b) is unbiased and consistent for $\mu$. (4 marks)

(e) Show that the estimator in (c) is biased and consistent for $\sigma^2$. (8 marks)

[Total: 25 marks]

**5.** (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of a test. (2 marks)

(ii) the Type II error of a test. (2 marks)

(iii) the significance level of a test. (2 marks)

(iv) the power function of a test (denoted $\Pi(\theta)$). (2 marks)

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with parameter $p$. State the rejection region for each of the following tests:

(i) $H_0 : p = p_0$ versus $H_1 : p \neq p_0$. (2 marks)

(ii) $H_0 : p = p_0$ versus $H_1 : p < p_0$. (2 marks)

In each case, assume a significance level of $\alpha$ and that $\overline{X} = \left(X_1 + X_2 + \cdots + X_n\right)/n$ has an approximate normal distribution.

(c) Under the same assumptions as in part (b), find the power function, $\Pi(p)$, for each of the tests:

(i) $H_0 : p = p_0$ versus $H_1 : p \neq p_0$. (5 marks)

(ii) $H_0 : p = p_0$ versus $H_1 : p < p_0$. (3 marks)

In each case, you should express the power function, $\Pi(p)$, in terms of $\Phi(\cdot)$, the standard normal distribution function.

(d) Are the power functions in (c) most powerful in the sense of the Neyman-Pearson lemma? Justify your answer. (5 marks)

[Total: 25 marks]

**6.** (a) Describe the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x; \theta)$. State both the test statistic and the rejection region. (4 marks)

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $f(x) = a\theta^a x^{-a-1}$, $x > \theta$, where $a$ is known but $\theta$ is unknown.

(i) Derive the Neyman-Pearson test for $H_0 : \theta = K$ versus $H_1 : \theta = L$, where $K > L$. Show that the rule for rejecting $H_0$ can be expressed as $\min(X_1, X_2, \ldots, X_n) < c$. (6 marks)

(ii) Show that the power function for the rejection rule in part (i) is

$$\Pi(\theta) = 1 - \frac{\theta^{na}}{c^{na}}.$$

(6 marks)

(iii) Find the value of $c$ if $n = 10$, $a = 1$, $K = 1$ and the probability of a type I error is 0.05. (4 marks)

(iv) Find the probability of a type II error if $n = 10$, $a = 1$, $K = 1$, $L = 1/2$ and the probability of a type I error is 0.05. (5 marks)

[Total: 25 marks]

**END OF EXAMINATION PAPER**