Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

15 May 2014

09:45-11:45

Answer FOUR of the SIX questions. If more than FOUR questions are attempted, then credit will be given for the best FOUR answers.

University-approved calculators may be used

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1. Let X denote an exponential random variable with its probability density function given by

$$f_X(x) = \lambda \exp(-\lambda x)$$

for $x > 0$ and $\lambda > 0$.

(i) Derive the moment generating function of X. (5 marks)

(ii) Use your result in (i) to derive the first four moments of X. (5 marks)

(iii) If X_i are independent and identical random variables and are distributed as X derive the moment generating function of $Y = X_1 + \cdots + X_n$. (6 marks)

(iv) Derive the mean and variance of Y. (6 marks)

(v) What is the distribution of Y? (3 marks)

[Total: 25 marks]

2. (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ; (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as bias$(\widehat{\theta})$); (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as MSE$(\widehat{\theta})$); (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ. (2 marks)

(b) Let X and Y be uncorrelated random variables. Suppose that X has mean 2θ and variance 4. Suppose that Y has mean θ and variance 2. The parameter θ is unknown.

(i) Compute the bias and mean squared error for each of the following estimators of θ: $\widehat{\theta_1} = (1/4)X + (1/2)Y$ and $\widehat{\theta_2} = X - Y$. (6 marks)

(ii) Which of the two estimators ($\widehat{\theta_1}$ or $\widehat{\theta_2}$) is better and why? (3 marks)

(iii) Verify that the estimator $\widehat{\theta_c} = (c/2)X + (1-c)Y$ is unbiased. Find the value of c which minimizes Var $\left(\widehat{\theta_c}\right)$. (6 marks)

[Total: 25 marks]

3. An electrical circuit consists of four batteries connected in series to a lightbulb. We model the battery lifetimes X_1, X_2, X_3, X_4 as independent and identically distributed $Uni(0, \theta)$ random variables. Our experiment to measure the operating time of the circuit is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min(X_1, X_2, X_3, X_4)$.

(i) Determine the cumulative distribution function of the random variable Y. (7 marks)

(ii) Write down the likelihood function of θ based on a single observation of Y. (4 marks)

(iii) Derive the maximum likelihood estimator of θ. (7 marks)

(iv) Find the bias of the estimator in part (iii). Is the estimator unbiased? (4 marks)

(v) Find the mean squared error of the estimator in part (iii). (3 marks)

[Total: 25 marks]

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(c\mu, \sigma^2\right)$, where both μ and σ^2 are unknown parameters and c is a fixed known constant.

(i) Write down the joint likelihood function of μ and σ^2. (5 marks)

(ii) Show that the maximum likelihood estimator (mle) of μ is $\widehat{\mu} = \overline{X}/c$, where $\overline{X} = (1/n) \sum_{i=1}^{n} X_i$ is the sample mean. (6 marks)

(iii) Show that the mle of σ^2 is $\widehat{\sigma^2} = (1/n) \sum_{i=1}^{n} \left(X_i - \overline{X}/c\right)^2$. (6 marks)

(iv) Show that the mle, $\widehat{\mu}$, is an unbiased and consistent estimator for μ. (4 marks)

(v) Show that the mle, $\widehat{\sigma^2}$, is a biased and consistent estimator for σ^2. (4 marks)

[Total: 25 marks]

5. (a) Suppose X and Y are independent normal random variables with means μ_X, μ_Y and common variance σ^2. Show that

$$\Pr(X < Y) = \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. (5 marks)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with unknown mean μ_X and known variance σ^2. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal population with unknown mean μ_Y and known variance σ^2. Assume independence of the two samples and let $p = \Pr(X_i < Y_i)$. Determine the power function, $\Pi(p)$, for testing the following hypotheses:

(i) $H_0 : p = 1/2$ versus $H_1 : p \neq 1/2$. (7 marks)

(ii) $H_0 : p \geq 1/2$ versus $H_1 : p < 1/2$. (7 marks)

(iii) $H_0 : p \leq 1/2$ versus $H_1 : p > 1/2$. (6 marks)

In each case, assume a significance level of α.

[Total: 25 marks]

6. (a) State the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x; \theta)$. (5 marks)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform$(0, \theta)$ distribution.

(i) Find the most powerful test at level α for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, where $\theta_2 < \theta_1$ are constants. Show that the test rejects H_0 if and only if $\max(X_1, X_2, \ldots, X_n) < k$ for some k. (6 marks)

(ii) Determine the power function, $\Pi(\theta)$, of the test in part (i). (6 marks)

(iii) Find the value of k when $\alpha = 0.05$, $n = 2$ and $\theta_1 = 0.2$. (4 marks)

(iv) Find $\beta = \Pr$ (Type II error) when $n = 2$, $\theta_1 = 0.2$ and $\theta_2 = 0.1$. (4 marks)

[Total: 25 marks]

END OF EXAMINATION PAPER