Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

Answer any FOUR of the questions.

University-approved calculators may be used

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1. This question explores the use of moment generating functions.

(i) Let X be a Laplace random variable with parameter a (denoted by L_a) and probability density function specified by

$$f(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right)$$

for $-\infty < x < \infty$ and $a > 0$. Show that the moment generating function of X is $1/(1 - a^2t^2)$. (7 marks)

(ii) Find the first four moments of L_a by differentiating the moment generating function. (7 marks)

(iii) If $X \sim L_a$ show that $\mid X \mid \sim Exp(1/a)$. (5 marks)

(iv) Let $X_1 \sim L_a$ and $X_2 \sim L_a$. Assume that X_1 and X_2 are independent. Find the moment generating function of $S = \mid X_1 \mid - \mid X_2 \mid$. (4 marks)

(v) Use the uniqueness of the moment generating function to determine the distribution of S. (2 marks)

[Total: 25 marks]

2. (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ; (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as $\text{bias}(\widehat{\theta})$); (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as $\text{MSE}(\widehat{\theta})$); (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ. (2 marks)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform $[0, \theta]$ distribution, where θ is unknown.

(i) Find the expected value and variance of the estimator $\widehat{\theta} = 2\overline{X}$, where $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$. (4 marks)

(ii) Find the expected value of the estimator $\max(X_1, X_2, \ldots, X_n)$, i.e. the largest observation. (4 marks)

(iii) Find the constant c such that $\widetilde{\theta} = c \max(X_1, X_2, \ldots, X_n)$ is an unbiased estimator of θ. Also find the variance of $\widetilde{\theta}$. (4 marks)

(iv) Compare the mean square errors of $\widehat{\theta}$ and $\widetilde{\theta}$ and comment. (3 marks)

[Total: 25 marks]

3. Consider the linear regression model with zero intercept:

$$Y_i = \beta X_i + e_i$$

for $i = 1, 2, \ldots, n$, where $e_1, e_2, \ldots, e_n$ are independent and identical normal random variables with zero mean and variance σ^2 assumed known. Moreover, suppose $X_1, X_2, \ldots, X_n$ are known constants.

(i) Write down the likelihood function of β. (5 marks)

(ii) Derive the maximum likelihood estimator of β. (6 marks)

(iii) Find the bias of the estimator in part (ii). Is the estimator unbiased? (5 marks)

(iv) Find the mean square error of the estimator in part (ii). (6 marks)

(v) Find the exact distribution of the estimator in part (ii). (3 marks)

[Total: 25 marks]

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the uniform $[\mu - \delta, \mu + \delta]$ distribution, where both μ and $\delta > 0$ are unknown.

(i) Write down the joint likelihood function of μ and δ. (9 marks)

(ii) Show that the maximum likelihood estimator (mle) of δ is $\widehat{\delta} = (1/2)\{\max(X_1, X_2, \ldots, X_n) - \min(X_1, X_2, \ldots, X_n)\}$. (2 marks)

(iii) Show that the mle of μ is $\widehat{\mu} = (1/2)\{\min(X_1, X_2, \ldots, X_n) + \max(X_1, X_2, \ldots, X_n)\}$. (2 marks)

(iv) Show that the mle $\widehat{\mu}$ is an unbiased estimator for μ. (9 marks)

(v) Show that the mle $\widehat{\delta}$ is a biased estimator for δ. Find an estimator $c\widehat{\delta}$ that is unbiased for δ. (3 marks)

[Total: 25 marks]

5. (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of a test. (2 marks)

(ii) the Type II error of a test. (2 marks)

(iii) the significance level of a test. (2 marks)

(iv) the power function of a test (denoted $\Pi(\theta)$). (2 marks)

(b) Let $x_1, x_2, \ldots, x_m$ be a random sample from a normal population with mean μ_X and variance σ_X^2 assumed known. Let $y_1, y_2, \ldots, y_n$ be a random sample from a normal population with mean μ_Y and variance σ_Y^2 assumed known. Assume independence of the two samples. Determine the power function, $\Pi(\mu_X, \mu_Y)$, for testing the following hypotheses:

(i) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$. (6 marks)

(ii) $H_0 : \mu_X \leq \mu_Y$ versus $H_1 : \mu_X > \mu_Y$. (6 marks)

(iii) $H_0 : \mu_X \geq \mu_Y$ versus $H_1 : \mu_X < \mu_Y$. (5 marks)

In each case, assume a significance level of α.

[Total: 25 marks]

6. (a) State the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x; \theta)$. (5 marks)

(b) Let X be one observation from a distribution specified by the pdf $f(x) = (1/\pi)[1 + (x - \theta)^2]^{-1}$ for $-\infty < x < \infty$.

(i) Find the most powerful test at level α for $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. (5 marks)

(ii) Show that the test in (i) rejects H_0 if and only if $X \in (a, b)$, where

$$a = \tan\left[\frac{1}{2}\arctan(-2) - \frac{\alpha}{2}\right]$$

and

$$b = \tan\left[\frac{1}{2}\arctan(-2) + \frac{\alpha}{2}\right].$$

(5 marks)

(iii) Find the type I error. (5 marks)

(iv) Find the type II error. (5 marks)

[Total: 25 marks]

END OF EXAMINATION PAPER