Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

Answer any FOUR of the questions.

University-approved calculators may be used

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1. This question explores the use of moment generating functions.

(i) Let X be a chi-square random variable with a degrees of freedom (denoted by χ^2_a) and probability density function specified by

$$f(x) = \frac{x^{a/2-1}\exp(-x/2)}{2^{a/2}\Gamma(a/2)}$$

for $x > 0$ and $a > 0$. Show that the moment generating function of X is $(1-2t)^{-a/2}$. (7 marks)

(ii) Find the first four moments of χ^2_a by differentiating the moment generating function. (7 marks)

(iii) Let $X_1 \sim \chi^2_a$, and $X_2 \sim \chi^2_b$. Assume that X_1 and X_2 are independent. Find the moment generating function of $S = X_1 + X_2$. (7 marks)

(iv) Use the linearity of expectation to find $E(S)$ and $Var(S)$. (2 marks)

(v) Use the uniqueness of the moment generating function to determine the distribution of S. (2 marks)

2. (a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ; (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ; (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as bias($\widehat{\theta}$)); (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as MSE($\widehat{\theta}$)); (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ. (2 marks)

(b) Let X and Y be uncorrelated random variables. Suppose that X has mean 2θ and variance 4. Suppose that Y has mean θ and variance 2. The parameter θ is unknown.

(i) Compute the bias and mean squared error for each of the following estimators of θ: $\widehat{\theta_1} = (1/4)X + (1/2)Y$ and $\widehat{\theta_2} = X - Y$. (6 marks)

(ii) Which of the two estimators ($\widehat{\theta_1}$ or $\widehat{\theta_2}$) is better and why? (3 marks)

(iii) Verify that the estimator $\widehat{\theta_c} = (c/2)X + (1-c)Y$ is unbiased. Find the value of c which minimizes Var ($\widehat{\theta_c}$). (6 marks)

3. An electrical circuit consists of three batteries connected in series to a lightbulb. We model the battery lifetimes X_1, X_2, X_3 as independent and identically distributed $Exp(\lambda)$ random variables. Our experiment to measure the operating time of the circuit is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min(X_1, X_2, X_3)$.

(i) Determine the distribution of the random variable Y. (7 marks)

(ii) Write down the likelihood function of λ. (4 marks)

(iii) Derive the maximum likelihood estimator of λ. (7 marks)

(iv) Find the bias of the estimator in part (iii). Is the estimator unbiased? (4 marks)

(v) Find the mean square error of the estimator in part (iii). (3 marks)

4. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with the common probability density function (pdf):

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}$$

for $x > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$. Both μ and σ^2 are unknown.

(i) Write down the joint likelihood function of μ and σ^2. (7 marks)

(ii) Show that the maximum likelihood estimator (mle) of μ is $\widehat{\mu} = (1/n)\sum_{i=1}^{n} \log X_i$. (2 marks)

(iii) Show that the mle of σ^2 is $\widehat{\sigma^2} = (1/n)\sum_{i=1}^{n}(\log X_i - \widehat{\mu})^2$. (2 marks)

(iv) Show that the mle $\widehat{\mu}$ is an unbiased and consistent estimator for μ. (7 marks)

(v) Show that the mle $\widehat{\sigma^2}$ is a biased and consistent estimator for σ^2. (7 marks)

For parts (iv) and (v), you may assume that $\log X_i$, $i = 1, 2, \ldots, n$ are independent and identically distributed normal random variables with mean μ and variance σ^2.

5. (a) Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of a test. (2 marks)

(ii) the Type II error of a test. (2 marks)

(iii) the significance level of a test. (2 marks)

(iv) the power function of a test (denoted $\Pi(\theta)$). (2 marks)

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, \sigma^2)$, where σ^2 is assumed known. Determine the power function, $\Pi(\theta)$, for testing the following hypotheses:

(i) $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. (6 marks)

(ii) $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. (6 marks)

(iii) $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. (5 marks)

In each case, assume a significance level of α.

6. (a) State the Neyman-Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x; \theta)$. (5 marks)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform$(0, \theta)$ distribution.

(i) Find the most powerful test at level α for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$, where $\theta_2 > \theta_1$ are constants. Show that the test rejects H_0 if and only if $\max(X_1, X_2, \ldots, X_n) > k$ for some k. (5 marks)

(ii) Determine the power function, $\Pi(\theta)$, of the test in part (i). (5 marks)

(iii) Find the value of k when $\alpha = 0.05$, $n = 5$ and $\theta_1 = 0.5$. (5 marks)

(iv) Find $\beta = \Pr$ (Type II error) when $n = 5$, $\theta_1 = 0.5$ and $\theta_2 = 0.6$. (5 marks)

END OF EXAMINATION PAPER