MATH20802

Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

21 June 2009

9:45 – 11:45

Answer any FOUR of the questions.

University-approved calculators may be used

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1. A random variable X is said to have the Gumbel distribution, written $X \sim \text{Gumbel}(\mu, \beta)$, if its probability density function is given by

$$f_X(x) = \frac{1}{\beta} \exp\left(-\frac{x-\mu}{\beta}\right) \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\beta > 0$.

(i) Show that the cumulative distribution function of X is

$$F_X(x) = \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\beta > 0$. (7 marks)

(ii) Show that the moment generating function of X is

$$M_X(t) = \exp(\mu t)\Gamma(1 - \beta t)$$

for $t < 1/\beta$, where $\Gamma(\cdot)$ denotes the gamma function. (7 marks)

(iii) Show that

$$E(X) = \mu - \beta\Gamma^{'}(1),$$

where $\Gamma^{'}(\cdot)$ denotes the first derivative of $\Gamma(\cdot)$. (4 marks)

(iv) If $X_i \sim \text{Gumbel}(\mu, \beta)$, $i = 1, 2, \ldots, n$ are independent random variables then shown that $\max(X_1, X_2, \ldots, X_n) \sim \text{Gumbel}(\mu + \beta \log n, \beta)$. (7 marks)

2. Suppose $\widehat{\theta}$ is an estimator of θ. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ. (2 marks)

(ii) $\widehat{\theta}$ is an asymptotically unbiased estimator of θ. (2 marks)

(iii) the bias of $\widehat{\theta}$ (written as bias $(\widehat{\theta})$). (2 marks)

(iv) the mean squared error of $\widehat{\theta}$ (written as MSE $(\widehat{\theta})$). (2 marks)

(v) $\widehat{\theta}$ is a consistent estimator of θ. (2 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Exp (λ) distribution. Consider the following estimators for $\theta = 1/\lambda$: $\widehat{\theta_1} = (1/n) \sum_{i=1}^n X_i$ and $\widehat{\theta_2} = (1/(n+1)) \sum_{i=1}^n X_i$.

(i) Find the biases of $\widehat{\theta_1}$ and $\widehat{\theta_2}$. (4 marks)

(ii) Find the variances of $\widehat{\theta_1}$ and $\widehat{\theta_2}$. (4 marks)

(iii) Find the mean squared errors of $\widehat{\theta_1}$ and $\widehat{\theta_2}$. (3 marks)

(iv) Which of the two estimators ($\widehat{\theta_1}$ or $\widehat{\theta_2}$) is better and why? (4 marks)

3. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with the common probability mass function (pmf):

$$p(x) = \theta(1-\theta)^{x-1}$$

for $x = 1, 2, \ldots$ and $0 < \theta < 1$.

(i) Write down the likelihood function of θ. (5 marks)

(ii) Find the maximum likelihood estimator (mle) of θ. (6 marks)

(iii) Find the mle of $\psi = 1/\theta$. (3 marks)

(iv) Determine the bias, variance and the mean squared error of the mle of ψ. (8 marks)

(v) Is the mle of ψ unbiased? Is it consistent? (3 marks)

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $Uni[a, b]$, where both a and b are unknown.

(i) Write down the joint likelihood function of a and b. (5 marks)

(ii) Show that the maximum likelihood estimator (mle) of a is $\widehat{a} = \min(X_1, X_2, \ldots, X_n)$. (5 marks)

(iii) Show that the mle of b is $\widehat{b} = \max(X_1, X_2, \ldots, X_n)$. (5 marks)

(iv) Show that the mle, $\widehat{a}$, is a biased and consistent estimator for a. (5 marks)

(v) Show that the mle, $\widehat{b}$, is also a biased and consistent estimator for b. (5 marks)

5. Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of the test. (2 marks)

(ii) the Type II error of the test. (2 marks)

(iii) the significance level of the test. (2 marks)

(iv) the power function of the test (denoted $\Pi(\theta)$). (2 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, \sigma^2)$, where σ is not known. Calculate the power function, $\Pi(\sigma)$, for each of the following tests:

(i) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$. (6 marks)

(ii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma < \sigma_0$. (6 marks)

(iii) $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$. (5 marks)

In each case, assume a significance level of α.

6. State the Neyman–Pearson test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x;\theta)$. (5 marks)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ distribution, where σ^2 is assumed known.

(i) Find the most powerful test at level α for $H_0 : \theta = \theta_1$ versus $\theta = \theta_2$, where $\theta_1 < \theta_2$ are constants. Show that the test rejects H_0 if and only if $(1/n)\sum_{i=1}^{n} X_i > k$ for some k. (5 marks)

(ii) Determine the power function, $\Pi(\theta)$, of the test in part (i). (5 marks)

(iii) Find the value of k when $\alpha = 0.01$, $n = 100$, $\sigma = 1$ and $\theta_1 = 1$. (5 marks)

(iv) Find $\beta = \Pr$ (Type II error) when $n = 100$, $\sigma = 1$ and $\theta_2 = 2$. (5 marks)

END OF EXAMINATION PAPER