Two hours

THE UNIVERSITY OF MANCHESTER

STATISTICAL METHODS

30 May 2008

2.00 – 4.00

Answer any FOUR of the questions.

Electronic calculators may be used, provided that they cannot store text.

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1. For a continuous random variable X let $M_X(t) = E[\exp(tX)]$ denote its moment generating function (mgf).

(i) A random variable X is said to the have Laplace distribution, written $X \sim Laplace(a, b)$, if its probability density function is given by

$$f(x) = \frac{1}{2b} \exp\left(-\frac{\mid x - a \mid}{b}\right)$$

for $-\infty < x < \infty$. Show that the mgf of $X \sim Laplace(a, b)$ is

$$M_X(t) = \frac{\exp(at)}{1 - b^2 t^2}$$

for $\mid t \mid< 1/b$. (7 marks)

(ii) If $X \sim Laplace(0, b)$ then show that $\mid X \mid\sim Exp(1/b)$. (6 marks)

(iii) If X_1 and X_2 are independent random variables then show that the mgf of $Y = aX_1 + bX_2$ can be written as $M_Y(t) = M_{X_1}(at) M_{X_2}(bt)$, where a and b are some constants. (6 marks)

(iv) If $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$ are independent random variables then show that $Y = \lambda_1 X_1 - \lambda_2 X_2 \sim Laplace(0, 1)$. (6 marks)

2. Suppose $\hat{\theta}$ is an estimator of θ. Define what is meant by the following:

(i) $\hat{\theta}$ is an unbiased estimator of θ. (2 marks)

(ii) $\hat{\theta}$ is an asymptotically unbiased estimator of θ. (2 marks)

(iii) the bias of $\hat{\theta}$ (written as bias $(\hat{\theta})$). (2 marks)

(iv) the mean squared error of $\hat{\theta}$ (written as MSE $(\hat{\theta})$). (2 marks)

(v) $\hat{\theta}$ is a consistent estimator of θ. (2 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the *Bernoulli*(p) distribution.

(i) What is the distribution of $\hat{p} = (1/n) \sum_{i=1}^{n} X_i$? Write down its probability mass function. (3 marks)

(ii) Show that $\hat{p}$ is unbiased and consistent for p. (4 marks)

(iii) If $\tilde{p} = (1/(n+1))(0.5 + \sum_{i=1}^{n} X_i)$ then show that $\tilde{p}$ is biased and consistent for p. (4 marks)

(iv) If $n = 100$ and $p = 0.5$ then which estimator ($\hat{p}$ or $\tilde{p}$) is the better estimator and why? (4 marks)

3. Consider the two independent random samples: $X_1, X_2, \ldots, X_n$ from $N(\mu_X, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_m$ from $N(\mu_Y, \sigma^2)$, where μ_X and μ_Y are assumed known.

(i) Write down the likelihood function of σ^2. (4 marks)

(ii) Find the maximum likelihood estimator (mle) of σ^2. (7 marks)

(iii) Show that the mle in part (ii) is an unbiased and consistent estimator for σ^2. (8 marks)

(iv) Find the mles of $\Pr(X_1 < Y_1)$ and $\Pr(X_1 + Y_1 < 1)$, where $X_1 \sim N(\mu_X, \sigma^2)$ and $Y_1 \sim N(\mu_Y, \sigma^2)$ are independent random variables. (6 marks)

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

(i) Write down the likelihood function of μ and σ^2. (5 marks)

(ii) Show that the maximum likelihood estimator (mle) of μ is $\hat{\mu} = \bar{X}$, where $\bar{X} = (1/n) \sum_{i=1}^{n} X_i$ is the sample mean. (5 marks)

(iii) Show that the mle of σ^2 is $\hat{\sigma^2} = (1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2$. (5 marks)

(iv) Show that the mle $\hat{\mu}$ is an unbiased and consistent estimator for μ. (5 marks)

(v) Show that the mle $\hat{\sigma^2}$ is a biased and consistent estimator for σ^2. (5 marks)

5. Suppose we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define what is meant by the following:

(i) the Type I error of the test. (2 marks)

(ii) the Type II error of the test. (2 marks)

(iii) the significance level of the test. (2 marks)

(iv) the power function of the test (denoted $\Pi(\theta)$). (2 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, \sigma^2)$, where σ^2 is assumed unknown. Calculate the power function, $\Pi(\theta)$, for each of the following tests:

(i) $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. (6 marks)

(ii) $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. (6 marks)

(iii) $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. (5 marks)

In each case, assume a significance level of α.

6. State the Neyman–Pearson lemma for testing $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ based on a random sample $X_1, X_2, \ldots, X_n$ from a distribution with the probability density function $f(x;\theta)$. (5 marks)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\theta, 1)$ distribution.

(i) Find the most powerful test at level α for $H_0 : \theta = \theta_1$ versus $\theta = \theta_2$, where $\theta_1 < \theta_2$ are constants. Show that the test rejects H_0 if and only if $(1/n)\sum_{i=1}^{n} X_i > k$ for some k. (10 marks)

(ii) Find the value of k when $\alpha = 0.01$, $n = 10$ and $\theta_1 = 1$. (5 marks)

(iii) Find $\beta = \Pr$ (Type II error) when $n = 10$ and $\theta_2 = 2$ in terms of the cdf of a standard normal random variable. (5 marks)

END OF EXAMINATION PAPER