MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

ANSWERS TO QUESTION 1

Let X denote an exponential random variable with its probability density function given by

$$f_X(x) = \exp(-x)$$

for $x > 0$.

(i) The moment generating function of X is

$$\begin{aligned} M_X(t) &= \int_0^\infty \exp\left[-(1-t)x\right] dx \\ &= \left[\frac{\exp\left[-(1-t)x\right]}{t-1}\right]_0^\infty \\ &= 0 - \frac{1}{t-1} \\ &= \frac{1}{1-t}. \end{aligned}$$

(ii) The first four derivatives of $M_X(t)$ are

$$\begin{aligned} M_X'(t) &= \frac{1}{(1-t)^2}, \\ M_X''(t) &= \frac{2}{(1-t)^3}, \\ M_X'''(t) &= \frac{6}{(1-t)^4}, \\ M_X''''(t) &= \frac{24}{(1-t)^5}, \end{aligned}$$

so

$$\begin{aligned} E(X) &= 1, \\ E\left(X^2\right) &= 2, \\ E\left(X^3\right) &= 6, \\ E\left(X^4\right) &= 24. \end{aligned}$$

(iii) The moment generating function of Y is

$$\begin{aligned} M_Y(t) &= E\left[\exp(tY)\right] \\ &= E\left[\exp\left(t\left(X_1 + \cdots + X_n\right)\right)\right] \\ &= E\left[\exp\left(tX_1\right) \cdots \exp\left(tX_n\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[\exp\left(tX_1\right)\right]\cdots E\left[\exp\left(tX_n\right)\right] \\
&= M_X(t)\cdots M_X(t) \\
&= M_X^n(t) \\
&= \frac{1}{(1-t)^n}.
\end{aligned}
$$

(iv) The mean and variance of Y are

$$E\left[X_1+\cdots+X_n\right] = E\left(X_1\right)+\cdots+E\left(X_n\right) = nE(X) = n$$

and

$$Var\left[X_1+\cdots+X_n\right] = Var\left(X_1\right)+\cdots+Var\left(X_n\right) = nVar(X) = n.$$

(v) Gamma distribution with parameters 1 and n.

ANSWERS TO QUESTION 2

An electrical circuit consists of four batteries connected in series to a lightbulb. We model the battery lifetimes X_1, X_2 as independent and identically distributed $Uni(0, \theta)$ random variables. Our experiment to measure the operating time of the circuit is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min(X_1, X_2)$.

(i) The cdf of Y is

$$\begin{aligned}\Pr(Y \leq y) &= 1 - \Pr[\min(X_1, X_2) > y] \\ &= 1 - \Pr(X_1 > y)\Pr(X_2 > y) \\ &= 1 - \Pr^2(X > y) \\ &= 1 - (1 - y/\theta)^2.\end{aligned}$$

(ii) The likelihood function of θ is

$$L(\theta) = 2(\theta - y)/\theta^2$$

for $0 < y < \theta$. The likelihood function of θ can also be written as

$$L(\theta) = \frac{2(\theta - y)}{\theta^2} I\{0 < y < \theta\}$$

for $\theta > 0$.

(iii) The MLE of θ can be found using both the standard and indicator function approaches.

Standard approach. The log-likelihood function is

$$\log L(\theta) = \log 2 + \log(\theta - y) - 2\log\theta$$

and

$$\frac{d \log L(\theta)}{d\theta} = \frac{1}{\theta - y} - \frac{2}{\theta}.$$

Setting $d \log L(\theta)/d\theta = 0$ gives $\widehat{\theta} = 2y$. This is an MLE since

$$\left.\frac{d^2 \log L(\theta)}{d\theta^2}\right|_{\widehat{\theta}=2y} = -\frac{1}{(\theta - y)^2} + \left.\frac{2}{\theta^2}\right|_{\widehat{\theta}=2y} = -\frac{1}{y^2} + \frac{1}{2y^2} = -\frac{1}{2y^2} < 0.$$

Indicator function approach. Plot $L(\theta) = \frac{2(\theta-y)}{\theta^2} I\{0 < y < \theta\}$ versus θ. The graph will be flat zero if $\theta \leq y$. When $\theta > y$,

$$L(\theta) = \frac{2(\theta - y)}{\theta^2}$$

and

$$\frac{dL(\theta)}{d\theta} = -\frac{2}{\theta^2} + \frac{4y}{\theta^3}$$

and setting this to zero given $\theta = 2y$. This solution corresponds to a maximum of the graph since

$$\left.\frac{d^2 L(\theta)}{d\theta^2}\right|_{\theta=2y} = \frac{4}{\theta^3} - \left.\frac{12y}{\theta^4}\right|_{\theta=2y} = \frac{1}{2y^3} - \frac{3}{4y^3} < 0.$$

Hence, the graph attains its maximum at $\theta = 2y$ and this is the MLE.

(iv) The bias of $\widehat{\theta}$ is

$$\begin{aligned}
Bias\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}\right) - \theta \\
&= \int_0^{\theta} 2y \frac{2(\theta - y)}{\theta^2} dy - \theta \\
&= \frac{4}{\theta^2} \int_0^{\theta} \left(\theta y - y^2\right) dy - \theta \\
&= \frac{4}{\theta^2} \left[\frac{\theta y^2}{2} - \frac{y^3}{3}\right]_0^{\theta} - \theta \\
&= \frac{4}{\theta^2} \left[\frac{\theta^3}{2} - \frac{\theta^3}{3}\right] - \theta \\
&= \frac{2\theta}{3} - \theta \\
&= -\frac{\theta}{3},
\end{aligned}$$

so the estimator is biased.

(v) The variance of $\widehat{\theta}$ is

$$\begin{aligned}
Var\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}^2\right) - E^2\left(\widehat{\theta}\right) \\
&= \int_0^{\theta} 4y^2 \frac{2(\theta - y)}{\theta^2} dy - \frac{4\theta^2}{9} \\
&= \frac{8}{\theta^2} \int_0^{\theta} \left(\theta y^2 - y^3\right) dy - \frac{4\theta^2}{9} \\
&= \frac{8}{\theta^2} \left[\frac{\theta y^3}{3} - \frac{y^4}{4}\right]_0^{\theta} - \frac{4\theta^2}{9} \\
&= \frac{8}{\theta^2} \left[\frac{\theta^4}{3} - \frac{\theta^4}{4}\right] - \frac{4\theta^2}{9} \\
&= \frac{2\theta^2}{3} - \frac{4\theta^2}{9} \\
&= \frac{2\theta^2}{9}.
\end{aligned}$$

So, the mean squared error of $\widehat{\lambda}$ is

$$MSE\left(\widehat{\theta}\right) = Var\left(\widehat{\theta}\right) + Bias^2\left(\widehat{\theta}\right) = \frac{2\theta^2}{9} + \frac{\theta^2}{9} = \frac{\theta^2}{3}.$$