# MATH20802: STATISTICAL METHODS
# SECOND SEMESTER
# ANSWERS TO THE IN CLASS TEST

## ANSWERS TO QUESTION 1

Let $X$ be a normal random variable with mean $\mu$ and standard deviation $\sigma$. Let $Y = \exp(2X)$; that is, $Y$ is a log-normal random variable with parameters $2\mu$ and $2\sigma$.

(i) If $X \sim N(\mu, \sigma^2)$ then

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^{\infty} \exp(tx) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + tx\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2\mu x - 2\sigma^2 tx + \mu^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 + \mu^2 - (\mu+\sigma^2 t)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma t)^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).
\end{aligned}$$

(ii) We have $E(Y) = M_X(2)$ since $E(Y) = E(\exp(2X))$. We have $E(Y^2) = M_X(4)$ since $E(Y^2) = E(\exp(4X))$.

(iii) From (ii), we have

$$E(Y) = M_X(2) = \exp\left(2\mu + 2\sigma^2\right)$$

and

$$E(Y^2) = M_X(4) = \exp\left(4\mu + 8\sigma^2\right)$$

So,

$$Var(Y) = \exp\left(4\mu + 8\sigma^2\right) - \exp\left(4\mu + 4\sigma^2\right).$$

(iv) We have

$$M_{X_1+X_2}(t) = E\left[\exp\left(tX_1 + tX_2\right)\right]$$

$$
\begin{aligned}
&= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\
&= \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) \exp\left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right) \\
&= \exp\left(\left(\mu_1 + \mu_2\right) t + \frac{\left(\sigma_1^2 + \sigma_2^2\right) t^2}{2}\right).
\end{aligned}
$$

(v) It follows from (iv) that $2X_1+2X_2$ is a normal random variable with mean $2\mu_1+2\mu_2$ and variance $4\sigma_1^2 + 4\sigma_2^2$. Hence, $\exp(2X_1 + 2X_2)$ is a log-normal random variable with parameter $2\mu_1 + 2\mu_2$ and $2\sqrt{\sigma_1^2 + \sigma_2^2}$.

**ANSWERS TO QUESTION 2**

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(c\mu, c\sigma^2\right)$, where both $\mu$ and $\sigma^2$ are unknown parameters and $c$ is a fixed known constant.

(i) The joint likelihood function of $\mu$ and $\sigma^2$ is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\sqrt{c}\sigma} \exp\left[ -\frac{\left(X_i - c\mu\right)^2}{2c\sigma^2} \right] \right\} \\ &= \frac{1}{(2\pi)^{n/2} c^{n/2} \sigma^n} \exp\left[ -\frac{1}{2c\sigma^2} \sum_{i=1}^{n} \left(X_i - c\mu\right)^2 \right]. \end{aligned}$$

The joint log likelihood function of $\mu$ and $\sigma^2$ is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log c - n\log\sigma - \frac{1}{2c\sigma^2}\sum_{i=1}^{n}\left(X_i - c\mu\right)^2.$$

The first order partial derivatives of this with respect to $\mu$ and $\sigma$ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - c\mu\right) = \frac{1}{\sigma^2}\left[\left(\sum_{i=1}^{n} X_i\right) - nc\mu\right] \tag{1}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{c\sigma^3}\sum_{i=1}^{n}\left(X_i - c\mu\right)^2, \tag{2}$$

respectively.

(ii) Using equation (1), one can see that the solution of $\partial \log L / \partial \mu = 0$ is $\mu = \overline{X}/c = (1/(nc))\sum_{i=1}^{n} X_i$.

(iii) Using equation (2), one can see that the solution of $\partial \log L / \partial \sigma = 0$ is $\sigma^2 = (1/(nc))\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$.

(iv) The mle, $\widehat{\mu}$, is an unbiased and consistent estimator for $\mu$ since

$$\begin{aligned} E\left(\widehat{\mu}\right) &= E\left(\frac{1}{nc}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{nc}\sum_{i=1}^{n} E\left(X_i\right) \\ &= \frac{1}{nc}\sum_{i=1}^{n} c\mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var\left(\widehat{\mu}\right) &= Var\left(\frac{1}{nc}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2c^2}\sum_{i=1}^{n} Var\left(X_i\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^2 c^2} \sum_{i=1}^{n} c\sigma^2 \\
&= \frac{\sigma^2}{nc}.
\end{aligned}
$$

(v) The mle, $\widehat{\sigma^2}$, is a biased and consistent estimator for $\sigma^2$ since

$$
\begin{aligned}
E\left(\widehat{\sigma^2}\right) &= E\left[\frac{1}{nc}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right] \\
&= E\left[\frac{n-1}{nc}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\frac{n-1}{c\sigma^2}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\chi^2_{n-1}\right] \\
&= \frac{(n-1)\sigma^2}{n}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(\widehat{\sigma^2}\right) &= Var\left[\frac{1}{nc}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right] \\
&= Var\left[\frac{n-1}{nc}S^2\right] \\
&= \frac{\sigma^4}{n^2}Var\left[\frac{n-1}{c\sigma^2}S^2\right] \\
&= \frac{\sigma^4}{n^2}Var\left[\chi^2_{n-1}\right] \\
&= \frac{2(n-1)\sigma^4}{n^2}.
\end{aligned}
$$

Note that we have used the fact $(n-1)S^2/\left(c\sigma^2\right) \sim \chi^2_{n-1}$. Furthermore, $S^2 = (1/(n-1))\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ denotes the sample variance.