MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

ANSWERS TO QUESTION 1

The following solution is correct if X has the pdf

$$f_X(x) = \frac{x^{a/2} \exp(-x/2)}{2^{a/2}\Gamma(a/2)}$$

for $x > 0$ and $a > 0$.

(i) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_0^\infty \exp(tx) \frac{x^{a/2} \exp(-x/2)}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{x^{a/2} \exp\left[-\left(1/2 - t\right) x\right]}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{y^{a/2} \exp\left(-y\right)}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} dx \qquad \text{substituting } y = \left(1/2 - t\right) x \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} \int_0^\infty y^{a/2} \exp\left(-y\right) dx \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} \Gamma(a/2+1) \qquad \text{by definition of gamma function} \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} (a/2)\Gamma(a/2) \qquad \text{by using the fact } \Gamma(s+1) = s\Gamma(s) \\
&= a \left(1 - 2t\right)^{-a/2-1}.
\end{aligned}$$

(ii) We have $E(Y) = M_X(1)$ since $E(Y) = E(\exp(X))$. We have $E(Y^2) = M_X(2)$ since $E(Y^2) = E(\exp(2X))$.

(iii) From (ii), we have

$$E(Y) = M_X(1) = a(-1)^{-a/2-1}$$

and

$$E(Y^2) = M_X(2) = a(-3)^{-a/2-1}.$$

So,

$$Var(Y) = a(-3)^{-a/2-1} - a^2(-1)^{-a-2}.$$

(iv) We have

$$\begin{aligned} M_{X_1+X_2}(t) &= E\left[\exp\left(tX_1 + tX_2\right)\right] \\ &= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\ &= a_1 a_2 \left(1 - 2t\right)^{-a_1/2 - a_2/2 - 2}. \end{aligned}$$

(v) It follows from (iv) that $X_1 + X_2$ is a gamma random variable with $\lambda = 1/2$ and $a = \left(a_1 + a_2\right)/2 + 2$ if $a_1 a_2 = 1$.

The following solution is correct if X has the pdf

$$f_X(x) = \frac{x^{a/2-1} \exp(-x/2)}{2^{a/2}\Gamma(a/2)}$$

for $x > 0$ and $a > 0$.

(i) The moment generating function of X is

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \int_0^\infty \exp(tx) \frac{x^{a/2-1} \exp(-x/2)}{2^{a/2}\Gamma(a/2)} dx \\ &= \int_0^\infty \frac{x^{a/2-1} \exp\left[-\left(1/2 - t\right) x\right]}{2^{a/2}\Gamma(a/2)} dx \\ &= \int_0^\infty \frac{y^{a/2-1} \exp\left(-y\right)}{2^{a/2}\Gamma(a/2)\left(1/2 - t\right)^{a/2}} dx \\ &= \frac{1}{2^{a/2}\Gamma(a/2)\left(1/2 - t\right)^{a/2}} \int_0^\infty y^{a/2-1} \exp\left(-y\right) dx \\ &= \frac{1}{2^{a/2}\Gamma(a/2)\left(1/2 - t\right)^{a/2}} \Gamma(a/2) \\ &= \frac{1}{2^{a/2}\left(1/2 - t\right)^{a/2}} \\ &= (1 - 2t)^{-a/2}. \end{aligned}$$

(ii) We have $E(Y) = M_X(1)$ since $E(Y) = E(\exp(X))$. We have $E(Y^2) = M_X(2)$ since $E(Y^2) = E(\exp(2X))$.

(iii) From (ii), we have

$$E(Y) = M_X(1) = (-1)^{a/2}$$

and

$$E(Y^2) = M_X(2) = (-3)^{a/2}.$$

So,

$$Var(Y) = (-3)^{a/2} - (-1)^a.$$

(iv) We have

$$\begin{aligned} M_{X_1+X_2}(t) &= E\left[\exp\left(tX_1 + tX_2\right)\right] \\ &= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\ &= (1-2t)^{-a_1/2}\,(1-2t)^{-a_2/2} \\ &= (1-2t)^{-(a_1+a_2)/2}\,. \end{aligned}$$

(v) It follows from (iv) that $X_1 + X_2$ is a chisquare random variable with degree of freedom equal to $a_1 + a_2$.

ANSWERS TO QUESTION 2

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, c\sigma^2\right)$, where both μ and σ^2 are unknown parameters and c is a fixed known constant.

(i) The joint likelihood function of μ and σ^2 is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\sqrt{c}\sigma} \exp\left[-\frac{\left(X_i - \mu\right)^2}{2c\sigma^2} \right] \right\} \\ &= \frac{1}{(2\pi)^{n/2} c^{n/2} \sigma^n} \exp\left[-\frac{1}{2c\sigma^2} \sum_{i=1}^{n} \left(X_i - \mu\right)^2 \right]. \end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log c - n\log\sigma - \frac{1}{2c\sigma^2} \sum_{i=1}^{n} \left(X_i - \mu\right)^2 .$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{c\sigma^2} \sum_{i=1}^{n} \left(X_i - \mu\right) = \frac{1}{c\sigma^2} \left[\left(\sum_{i=1}^{n} X_i \right) - n\mu \right] \tag{1}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{c\sigma^3} \sum_{i=1}^{n} \left(X_i - \mu\right)^2 , \tag{2}$$

respectively.

(ii) Using equation (1), one can see that the solution of $\partial \log L / \partial \mu = 0$ is $\mu = \overline{X} = (1/n)\sum_{i=1}^{n} X_i$.

(iii) Using equation (2), one can see that the solution of $\partial \log L / \partial \sigma = 0$ is $\sigma^2 = (1/(nc))\sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$.

(iv) The mle, $\widehat{\mu}$, is an unbiased and consistent estimator for μ since

$$\begin{aligned} E\left(\widehat{\mu}\right) &= E\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) \\ &= \frac{1}{n} \sum_{i=1}^{n} E\left(X_i\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var\left(\widehat{\mu}\right) &= Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} Var\left(X_i\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i=1}^{n} c\sigma^2 \\
&= \frac{c\sigma^2}{n}.
\end{aligned}
$$

(v) The mle, $\widehat{\sigma^2}$, is a biased and consistent estimator for σ^2 since

$$
\begin{aligned}
E\left(\widehat{\sigma^2}\right) &= E\left[\frac{1}{nc}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right] \\
&= E\left[\frac{n-1}{nc}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\frac{n-1}{c\sigma^2}S^2\right] \\
&= \frac{\sigma^2}{n}E\left[\chi^2_{n-1}\right] \\
&= \frac{(n-1)\sigma^2}{n}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(\widehat{\sigma^2}\right) &= Var\left[\frac{1}{nc}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right] \\
&= Var\left[\frac{n-1}{nc}S^2\right] \\
&= \frac{\sigma^4}{n^2}Var\left[\frac{n-1}{c\sigma^2}S^2\right] \\
&= \frac{\sigma^4}{n^2}Var\left[\chi^2_{n-1}\right] \\
&= \frac{2(n-1)\sigma^4}{n^2}.
\end{aligned}
$$

Note that we have used the fact $(n-1)S^2/\left(c\sigma^2\right) \sim \chi^2_{n-1}$. Furthermore, $S^2 = (1/(n-1))\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ denotes the sample variance.