MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

ANSWERS TO QUESTION 1

Let X be a normal random variable with mean μ and standard deviation σ. Let $Y = \exp(X)$; that is, Y is a log-normal random variable with parameters μ and σ.

(i) If $X \sim N(\mu, \sigma^2)$ then

$$
\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^{\infty} \exp(tx) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + tx\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2\mu x - 2\sigma^2 tx + \mu^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 + \mu^2 - (\mu+\sigma^2 t)^2}{2\sigma^2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\mu-\sigma t)^2}{2\sigma^2}\right\} dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).
\end{aligned}
$$

(ii) We have $E(Y) = M_X(1)$ since $E(Y) = E(\exp(X))$. We have $E(Y^2) = M_X(2)$ since $E(Y^2) = E(\exp(2X))$.

(iii) From (ii), we have

$$E(Y) = M_X(1) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

and

$$E(Y^2) = M_X(2) = \exp\left(2\mu + 2\sigma^2\right)$$

So,

$$Var(Y) = \exp\left(2\mu + 2\sigma^2\right) - \exp\left(2\mu + \sigma^2\right).$$

(iv) We have

$$M_{X_1+X_2}(t) = E\left[\exp\left(tX_1 + tX_2\right)\right]$$

$$
\begin{aligned}
&= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\
&= \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) \exp\left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right) \\
&= \exp\left(\left(\mu_1 + \mu_2\right) t + \frac{\left(\sigma_1^2 + \sigma_2^2\right) t^2}{2}\right).
\end{aligned}
$$

(v) It follows from (iv) that $X_1 + X_2$ is a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Hence, $\exp(X_1 + X_2)$ is a log-normal random variable with parameter $\mu_1 + \mu_2$ and $\sqrt{\sigma_1^2 + \sigma_2^2}$.

ANSWERS TO QUESTION 2

Consider the linear regression model with zero intercept:

$$Y_i = \beta X_i + e_i$$

for $i = 1, 2, \ldots, n$, where $e_1, e_2, \ldots, e_n$ are independent and identical normal random variables with zero mean and variance σ^2 assumed known. Moreover, suppose $X_1, X_2, \ldots, X_n$ are known constants.

(i) The likelihood function of β is

$$L(\beta) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{ -\sum_{i=1}^{n} \frac{e_i^2}{2\sigma^2} \right\} = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{ -\sum_{i=1}^{n} \frac{(Y_i - \beta X_i)^2}{2\sigma^2} \right\}.$$

(ii) The log likelihood function of β is

$$\log L(\beta) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n} (Y_i - \beta X_i)^2 .$$

The normal equation is:

$$\frac{\partial \log L(\beta)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^{n} (Y_i - \beta X_i) X_i = 0.$$

Solving this equation gives

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}.$$

This is an mle since

$$\frac{\partial^2 \log L(\beta)}{\partial \beta^2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2 < 0.$$

(iii) The bias of $\widehat{\beta}$ is

$$\begin{aligned} E\widehat{\beta} - \beta &= E\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} - \beta \\ &= \frac{\sum_{i=1}^{n} X_i E(Y_i)}{\sum_{i=1}^{n} X_i^2} - \beta \\ &= \frac{\sum_{i=1}^{n} X_i \beta X_i}{\sum_{i=1}^{n} X_i^2} - \beta \\ &= 0, \end{aligned}$$

so $\widehat{\beta}$ is indeed unbiased.

(iv) The variance of $\widehat{\beta}$ is

$$\begin{aligned} Var\widehat{\beta} &= Var\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} \\ &= \frac{\sum_{i=1}^{n} X_i^2 Var\left(Y_i\right)}{\left(\sum_{i=1}^{n} X_i^2\right)^2} \\ &= \frac{\sum_{i=1}^{n} X_i^2 \sigma^2}{\left(\sum_{i=1}^{n} X_i^2\right)^2} \\ &= \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}. \end{aligned}$$

(v) The estimator is a linear combination of independent normal random variable, so it is also normal with mean β and variance $\frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}$.