MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

1. For a continuous random random variable X let $M_X(t) = E[\exp(tX)]$, $t > 0$ denote its moment generating function (mgf).

   (i) $X \sim Geom(\theta)$ with the pmf $p(x) = \theta(1-\theta)^{x-1}$ for $x = 1, 2, \ldots$ then

$$\begin{aligned} M_X(t) &= \sum_{x=1}^{\infty} \exp(tx)\theta(1-\theta)^{x-1} \\ &= \theta \exp(t) \sum_{x=1}^{\infty} \exp(t(x-1))(1-\theta)^{x-1} \\ &= \theta \exp(t) \sum_{y=0}^{\infty} \exp(ty)(1-\theta)^{y} \\ &= \frac{\theta \exp(t)}{1-\exp(t)(1-\theta)}. \end{aligned}$$

   (ii) if $X \sim Exp(\lambda)$ with the pdf $f(x) = \lambda \exp(-\lambda x)$ for $x > 0$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \lambda \int_0^{\infty} \exp(tx - \lambda x) dx \\ &= \frac{\lambda}{\lambda - t}\left[\exp(-(\lambda - t)x)\right]_0^{\infty} \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

   provided that $\lambda - t > 0$.

   (iii) if X is a continuous random variable with the pdf $f(x) = \exp(-x)/\{1+\exp(-x)\}^2$ for $-\infty < x < \infty$ then

$$M_X(t) = E\left[\exp(tX)\right] = \int_{-\infty}^{\infty} \frac{\exp(tx - x)}{\left[1+\exp(-x)\right]^2} dx. \tag{1}$$

   Set $y = 1/\left\{1+\exp(-x)\right\}$, so

$$\begin{aligned} & 1+\exp(-x) = \frac{1}{y} \\ \Longrightarrow\ & \exp(-x) = \frac{1}{y} - 1 = \frac{1-y}{y} \\ \Longrightarrow\ & x = -\log\left(\frac{1-y}{y}\right) \\ \Longrightarrow\ & x = \log y - \log(1-y) \end{aligned}$$

and

$$\frac{dx}{dy} = \frac{d \log y}{dy} - \frac{d \log(1-y)}{dy} = \frac{1}{y} + \frac{1}{1-y} = \frac{1}{y(1-y)}.$$

Substituting these into (1) gives

$$\begin{aligned} M_X(t) &= \int_0^1 [\exp(-x)]^{1-t} \frac{1}{[1+\exp(-x)]^2} \frac{dx}{dy} dy \\ &= \int_0^1 \left[\frac{1-y}{y} \right]^{1-t} y^2 \frac{1}{y(1-y)} dy \\ &= \int_0^1 (1-y)^{-t} y^t dy \\ &= B(1-t, 1+t). \end{aligned}$$

(iv) if X is a continuous random variable with the pdf

$$f(x) = \begin{cases} 4x, & \text{if } 0 < x < 1/2, \\ 4(1-x), & \text{if } 1/2 \leq x < 1 \end{cases}$$

then

$$\begin{aligned} M_X(t) &= E[\exp(tX)] \\ &= 4 \left\{ \int_0^{1/2} x \exp(tx) dx + \int_{1/2}^1 (1-x) \exp(tx) dx \right\} \\ &= 4 \left\{ \left[x \frac{\exp(tx)}{t} \right]_0^{1/2} - \frac{1}{t} \int_0^{1/2} \exp(tx) dx + \left[(1-x) \frac{\exp(tx)}{t} \right]_{1/2}^1 + \frac{1}{t} \int_{1/2}^1 \exp(tx) dx \right\} \\ &= 4 \left\{ \frac{\exp(t/2)}{2t} - \frac{\exp(t/2)-1}{t^2} - \frac{\exp(t/2)}{2t} + \frac{\exp(t)-\exp(t/2)}{t^2} \right\} \\ &= 4 \frac{\exp(t) - 2\exp(t/2) + 1}{t^2}. \end{aligned}$$

(v) if X is a continuous random variable X with the pdf $f(x) = (1/x^2) \exp(-1/x)$ for $x > 0$ then

$$M_X(t) = E[\exp(tX)] = \int_0^\infty \left(1/x^2\right) \exp\left(-1/x + tx\right) dx = \infty,$$

so the mgf is undefined.

2. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with the common probability density function (pdf):

$$f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$$

for $0 < x < \theta_1$, $\theta_1 > 0$ and $\theta_2 > 0$. Both θ_1 and θ_2 are unknown.

(i) The cumulative distribution function corresponding to the given pdf is

$$F(x) = \theta_2 \theta_1^{-\theta_2} \int_0^x y^{\theta_2 - 1} dy = \theta_1^{-\theta_2} x^{\theta_2}.$$

The mean corresponding to the given pdf is

$$E(X) = \theta_2 \theta_1^{-\theta_2} \int_0^{\theta_1} y^{\theta_2} dy = \frac{\theta_1 \theta_2}{\theta_2 + 1}.$$

The variance corresponding to the given pdf is

$$Var(X) = \theta_2 \theta_1^{-\theta_2} \int_0^{\theta_1} y^{\theta_2 + 1} dy - \frac{\theta_1^2 \theta_2^2}{(\theta_2 + 1)^2} = \frac{\theta_1^2 \theta_2}{\theta_2 + 2} - \frac{\theta_1^2 \theta_2^2}{(\theta_2 + 1)^2}.$$

(ii) The joint likelihood function of θ_1 and θ_2 is

$$\begin{aligned} L(\theta_1, \theta_2) &= \prod_{i=1}^n \left[\theta_2 x_i^{\theta_2 - 1} \theta_1^{-\theta_2} I\{0 < x_i < \theta_1\} \right] \\ &= \theta_2^n \left(\prod_{i=1}^n x_i \right)^{\theta_2 - 1} \theta_1^{-n\theta_2} \left(\prod_{i=1}^n I\{0 < x_i < \theta_1\} \right) \\ &= \theta_2^n \left(\prod_{i=1}^n x_i \right)^{\theta_2 - 1} \theta_1^{-n\theta_2} I\{\max(x_1, x_2, \ldots, x_n) < \theta_1\} \end{aligned}$$

for $\theta_1 > 0$ and $\theta_2 > 0$.

(iii) The likelihood function monotonically decreases with respect to θ_1. The lowest possible value for θ_1 is $\max(X_1, X_2, \ldots, X_n)$. So, the mle of θ_1 is $\max(X_1, X_2, \ldots, X_n)$.

(iv) The log of the joint likelihood function is

$$\log L(\theta_1, \theta_2) = n \log \theta_2 - n\theta_2 \log \theta_1 + (\theta_2 - 1) \sum_{i=1}^n \log x_i.$$

The first derivative of the log likelihood with respect to θ_2 is

$$\frac{d \log L(\theta_1, \theta_2)}{d\theta_2} = \frac{n}{\theta_2} - n \log \theta_1 + \sum_{i=1}^n \log x_i.$$

Setting this to zero and solving, we obtain $\widehat{\theta_2} = n / \{n \log \widehat{\theta_1} - \sum_{i=1}^n \log x_i\}$, where $\widehat{\theta_1} = \max(X_1, X_2, \ldots, X_n)$. The second derivative of the log likelihood with respect to θ_2

$$\frac{d^2 \log L(\theta_1, \theta_2)}{d\theta_2^2} = -\frac{n}{\theta_2^2} < 0.$$

Hence, $\widehat{\theta_2} = n / \{n \log \widehat{\theta_1} - \sum_{i=1}^n \log x_i\}$ is indeed the mle of θ_2.

(v) Let $Z = \max(X_1, X_2, \ldots, X_n)$. Then the cdf of Z is

$$\begin{aligned} \Pr(Z < z) &= \Pr(\max(X_1, X_2, \ldots, X_n) < z) \\ &= \Pr(X_1 < z, X_2 < z, \ldots, X_n < z) \\ &= \Pr^n(X < z) \\ &= \theta_1^{-n\theta_2} x^{n\theta_2}. \end{aligned}$$

It follows that Z has the same distribution as the given pdf with θ_2 replaced by $n\theta_2$. So, the bias $\widehat{\theta_1}$ is

$$E\left(\widehat{\theta_1}\right) - \theta_1 = \frac{n\theta_1\theta_2}{n\theta_2 + 1} - \theta_1 = -\frac{\theta_1}{n\theta_2 + 1} < 0.$$

The variance $\widehat{\theta_1}$ is

$$Var\left(\widehat{\theta_1}\right) = \frac{n\theta_1^2\theta_2}{n\theta_2 + 2} - \frac{n^2\theta_1^2\theta_2^2}{(n\theta_2 + 1)^2}$$

and the MSE $\widehat{\theta_1}$ is

$$MSE\left(\widehat{\theta_1}\right) = \frac{\theta_1^2}{(n\theta_2 + 1)^2} + \frac{n\theta_1^2\theta_2}{n\theta_2 + 2} - \frac{n^2\theta_1^2\theta_2^2}{(n\theta_2 + 1)^2}.$$

The MSE approaches zero as $n \to \infty$. Hence, $\widehat{\theta_1}$ is a biased and consistent estimator for θ_1.