MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

1. For a random variable X let $M_X(t) = E[\exp(tX)]$ denote its moment generating function (mgf).

   (i) if $X \sim Bin(n, p)$ with the pmf $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \ldots, n$ and $0 < p < 1$ then

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{n} \exp(tx) \binom{n}{x} p^x (1-p)^{n-x} \\
&= (1-p)^n \sum_{x=0}^{n} \binom{n}{x} \left\{ \frac{p\exp(t)}{1-p} \right\}^x \\
&= (1-p)^n \left\{ 1 + \frac{p\exp(t)}{1-p} \right\}^n \\
&= \{1 - p + p\exp(t)\}^n .
\end{aligned}
$$

   (ii) if $X \sim Po(\theta)$ with the pmf $p(x) = \theta^x \exp(-\theta)/x!$ for $x = 0, 1, \ldots$ then

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} \exp(tx) \frac{\theta^x \exp(-\theta)}{x!} \\
&= \exp(-\theta) \sum_{x=0}^{\infty} \frac{1}{x!} (\theta \exp(t))^x \\
&= \exp(-\theta) \exp(\theta \exp(t)) \\
&= \exp\{\theta [\exp(t) - 1]\} .
\end{aligned}
$$

   (iii) if $X \sim Cauchy(0, 1)$ with the pdf $f(x) = (1/\pi)(1 + x^2)^{-1}$ for $-\infty < x < \infty$ then

$$
M_X(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(tx)}{1 + x^2} dx = 1
$$

   if $t = 0$;

$$
\begin{aligned}
M_X(t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(tx)}{1 + x^2} dx \\
&\geq \frac{1}{\pi} \int_{0}^{\infty} \frac{\exp(tx)}{1 + x^2} dx \\
&\geq \frac{t}{2\pi} \int_{0}^{\infty} \frac{2x}{1 + x^2} dx \\
&= \frac{t}{2\pi} \left[\log\left(1 + x^2\right) \right]_0^{\infty} \\
&= \infty
\end{aligned}
$$

   if $t > 0$; and,

$$
M_X(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(tx)}{1 + x^2} dx
$$

$$
\begin{aligned}
&\geq \frac{1}{\pi}\int_{-\infty}^{0}\frac{\exp(tx)}{1+x^2}dx \\
&\geq \frac{t}{2\pi}\int_{-\infty}^{0}\frac{2x}{1+x^2}dx \\
&= \frac{t}{2\pi}\left[\log\left(1+x^2\right)\right]_{-\infty}^{0} \\
&= \infty
\end{aligned}
$$

if $t < 0$.

(iv) if $X \sim Gamma(a,b)$ with the pdf $f(x) = b^a x^{a-1}\exp(-bx)/\Gamma(a)$ for $x > 0$, $a > 0$ and $b > 0$, where $\Gamma(\cdot)$ denotes the gamma function, then

$$
\begin{aligned}
M_X(t) &= \frac{b^a}{\Gamma(a)}\int_0^{\infty} x^{a-1}\exp\{-(b-t)x\}\, dx \\
&= \frac{b^a}{(b-t)^a\Gamma(a)}\int_0^{\infty} y^{a-1}\exp\{-y\}\, dy \\
&= \frac{b^a}{(b-t)^a\Gamma(a)}\Gamma(a) \\
&= \left(\frac{b}{b-t}\right)^a.
\end{aligned}
$$

(v) if $X \sim Beta(3,1)$ with the pdf $f(x) = 3x^2$ for $0 < x < 1$ then

$$
\begin{aligned}
M_X(t) &= 3\int_0^1 x^2\exp(tx)dx \\
&= 3\left\{\left[\frac{x^2\exp(tx)}{t}\right]_0^1 - \frac{2}{t}\int_0^1 x\exp(tx)dx\right\} \\
&= 3\left\{\frac{\exp(t)}{t} - \frac{2}{t}\left\{\left[\frac{x\exp(tx)}{t}\right]_0^1 - \frac{1}{t}\int_0^1 \exp(tx)dx\right\}\right\} \\
&= 3\left\{\frac{\exp(t)}{t} - \frac{2\exp(t)}{t^2} + \frac{2}{t^2}\int_0^1 \exp(tx)dx\right\} \\
&= 3\left\{\frac{\exp(t)}{t} - \frac{2\exp(t)}{t^2} + \frac{2\left[\exp(t)-1\right]}{t^3}\right\}.
\end{aligned}
$$

2. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $Uni[a,b]$, where both a and b are unknown.

(i) The joint likelihood function of a and b is

$$
\begin{aligned}
L(a,b) &= \frac{1}{b-a}I\{a < X_1 < b\}\frac{1}{b-a}I\{a < X_2 < b\}\cdots\frac{1}{b-a}I\{a < X_n < b\} \\
&= \frac{1}{(b-a)^n}\prod_{i=1}^{n} I\{a < X_i < b\} \\
&= \frac{1}{(b-a)^n}I\{\max(X_1, X_2, \ldots, X_n) < b\}I\{a < \min(X_1, X_2, \ldots, X_n)\},
\end{aligned}
$$

where $I\{\cdot\}$ denotes the indicator function.

(ii) Note that $(b-a)^{-n}$ is an increasing function of a over $(-\infty, \min(X_1, X_2, \ldots, X_n))$. So, the maximum of $L(a,b)$ will be attained at $a = \min(X_1, X_2, \ldots, X_n)$.

(iii) Note that $(b-a)^{-n}$ is a decreasing function of b over $(\max(X_1, X_2, \ldots, X_n), \infty)$. So, the maximum of $L(a,b)$ will be attained at $b = \max(X_1, X_2, \ldots, X_n)$.

(iv) Let $Z = \min(X_1, X_2, \ldots, X_n)$. Note that

$$F_Z(z) = 1 - \left(\frac{b-z}{b-a} \right)^n$$

and

$$f_Z(z) = n \frac{(b-z)^{n-1}}{(b-a)^n}$$

for $a < z < b$. Furthermore,

$$\begin{aligned} E(Z) &= n \int_a^b z \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= n \int_a^b (b-(b-z)) \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= nb \int_a^b \frac{(b-z)^{n-1}}{(b-a)^n} dz - n \int_a^b \frac{(b-z)^n}{(b-a)^n} dz \\ &= b - \frac{n(b-a)}{n+1} \\ &= \frac{na}{n+1} + \frac{b}{n+1}, \end{aligned}$$

$$\begin{aligned} E\left(Z^2\right) &= n \int_a^b z^2 \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= n \int_a^b (b-(b-z))^2 \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= nb^2 \int_a^b \frac{(b-z)^{n-1}}{(b-a)^n} dz - 2nb \int_a^b \frac{(b-z)^n}{(b-a)^n} dz + n \int_a^b \frac{(b-z)^{n+1}}{(b-a)^n} dz \\ &= b^2 - \frac{2nb(b-a)}{n+1} + \frac{n(b-a)^2}{n+2}, \end{aligned}$$

and

$$Var\left(Z\right) = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2} \right] (b-a)^2.$$

So, it follows that the mle, $\hat{a}$, is a biased and consistent estimator for a.

(v) Let $Z = \max(X_1, X_2, \ldots, X_n)$. Note that

$$F_Z(z) = \left(\frac{z-a}{b-a} \right)^n$$

and

$$f_Z(z) = n \frac{(z-a)^{n-1}}{(b-a)^n}$$

for $a < z < b$. Furthermore,

$$\begin{aligned} E(Z) &= n\int_a^b z\frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= n\int_a^b (z-a+a)\frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= n\int_a^b \frac{(z-a)^{n}}{(b-a)^n}dz + na\int_a^b \frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= \frac{n(b-a)}{n+1} + a \\ &= \frac{a}{n+1} + \frac{nb}{n+1}, \end{aligned}$$

$$\begin{aligned} E\left(Z^2\right) &= n\int_a^b z^2\frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= n\int_a^b (z-a+a)^2\frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= n\int_a^b \frac{(z-a)^{n+1}}{(b-a)^n}dz + 2na\int_a^b \frac{(z-a)^{n}}{(b-a)^n}dz + na^2\int_a^b \frac{(z-a)^{n-1}}{(b-a)^n}dz \\ &= \frac{n(b-a)^2}{n+2} + \frac{2na(b-a)}{n+1} + a^2, \end{aligned}$$

and

$$Var\left(Z\right) = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right](b-a)^2.$$

So, it follows that the mle, $\hat{b}$, is a biased and consistent estimator for b.