MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

1. For a continuous random random variable X let $M_X(t) = E[\exp(tX)]$ denote its moment generating function (mgf).

   (i) $X \sim Geom(\theta)$ with the pmf $p(x) = \theta(1-\theta)^{x-1}$ for $x = 1, 2, \ldots$ then

$$
\begin{aligned}
M_X(t) &= \sum_{x=1}^{\infty} \exp(tx)\theta(1-\theta)^{x-1} \\
&= \theta \exp(t) \sum_{x=1}^{\infty} \exp(t(x-1))(1-\theta)^{x-1} \\
&= \theta \exp(t) \sum_{y=0}^{\infty} \exp(ty)(1-\theta)^{y} \\
&= \frac{\theta \exp(t)}{1-\exp(t)(1-\theta)}.
\end{aligned}
$$

   (ii) $X \sim Po(\theta)$ with the pmf $p(x) = \theta^x \exp(-\theta)/x!$ for $x = 0, 1, \ldots$ then

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} \exp(tx)\frac{\theta^x \exp(-\theta)}{x!} \\
&= \exp(-\theta) \sum_{x=0}^{\infty} \frac{1}{x!}\left(\theta \exp(t)\right)^x \\
&= \exp(-\theta)\exp\left(\theta \exp(t)\right) \\
&= \exp\left\{\theta\left[\exp(t) - 1\right]\right\}.
\end{aligned}
$$

   (iii) if $X \sim Laplace(a, b)$ with the pdf $f(x) = (1/(2b))\exp(- \mid x-a \mid /b)$ for $-\infty < x < \infty$ then

$$
\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \frac{1}{2b}\int_{-\infty}^{\infty} \exp(tx)\exp\left(-\frac{\mid x-a \mid}{b}\right) dx \\
&= \frac{1}{2b}\left[\int_a^{\infty} \exp\left(tx - \frac{x-a}{b}\right) dx + \int_{-\infty}^{a} \exp\left(tx + \frac{x-a}{b}\right) dx\right] \\
&= \frac{1}{2b}\left[\exp\left(\frac{a}{b}\right)\int_a^{\infty} \exp\left\{\left(t-\frac{1}{b}\right)x\right\} dx + \exp\left(-\frac{a}{b}\right)\int_{-\infty}^{a} \exp\left\{\left(t+\frac{1}{b}\right)x\right\} dx\right] \\
&= \frac{1}{2b}\left[\exp\left(\frac{a}{b}\right)\frac{1}{1/b-t}\exp\left\{\left(t-\frac{1}{b}\right)a\right\} + \exp\left(-\frac{a}{b}\right)\frac{1}{1/b+t}\exp\left\{\left(t+\frac{1}{b}\right)a\right\}\right] \\
&= \frac{1}{2}\exp(at)\left[\frac{1}{1-bt} + \frac{1}{1+bt}\right] \\
&= \frac{\exp(at)}{1-b^2t^2}.
\end{aligned}
$$

(iv) if $X \sim Exp(\lambda)$ with the pdf $f(x) = \lambda \exp(-\lambda x)$ for $x > 0$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \lambda \int_0^\infty \exp(tx - \lambda x) dx \\ &= \frac{\lambda}{\lambda - t} \left[\exp(-(\lambda - t)x)\right]_0^\infty \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

provided that $\lambda - t > 0$.

(v) if $X \sim Gumbel(\mu, \beta)$ with the pdf $f(x) = (1/\beta)\exp[-(x-\mu)/\beta]\exp\{-\exp[-(x-\mu)/\beta]\}$ for $-\infty < x < \infty$ then

$$\begin{aligned} M_X(t) &= E\left[\exp(tX)\right] \\ &= \int_{-\infty}^\infty \frac{1}{\beta} \exp\left(tx - \frac{x-\mu}{\beta}\right) \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\} dx \\ &= \int_0^\infty \frac{1}{\beta} \exp\left\{t(\mu - \beta \log z)\right\} z \exp\left\{-z\right\} \frac{\beta}{z} dz \\ &= \exp(\mu t) \int_0^\infty \exp\left\{-t\beta \log z\right\} \exp\left\{-z\right\} dz \\ &= \exp(\mu t) \int_0^\infty z^{-\beta t} \exp\left\{-z\right\} dz \\ &= \exp(\mu t)\Gamma(1 - \beta t), \end{aligned}$$

where the last step follows by the definition of the gamma function.

2. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

(i) The joint likelihood function of μ and σ^2 is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^n \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(X_i - \mu\right)^2}{2\sigma^2}\right] \right\} \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right)^2\right]. \end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right)^2.$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right) = \frac{1}{\sigma^2} \left(\sum_{i=1}^n X_i - n\mu \right) \tag{1}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n \left(X_i - \mu\right)^2, \tag{2}$$

respectively.

(ii) Using equation (1), one can see that the solution of $\partial \log L / \partial \mu = 0$ is $\mu = \bar{X} = (1/n) \sum_{i=1}^{n} X_i$.

(iii) Using equation (2), one can see that the solution of $\partial \log L / \partial \sigma = 0$ is $\sigma^2 = (1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2$.

(iv) The mle, $\widehat{\mu}$, is an unbiased and consistent estimator for μ since

$$
\begin{aligned}
E\left(\widehat{\mu}\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu \\
&= \mu
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(\widehat{\mu}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

(v) The mle, $\widehat{\sigma^2}$, is a biased and consistent estimator for σ^2 since

$$
\begin{aligned}
E\left(\widehat{\sigma^2}\right) &= E\left[\frac{1}{n}\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2\right] \\
&= E\left[\frac{n-1}{n} S^2\right] \\
&= \frac{\sigma^2}{n} E\left[\frac{n-1}{\sigma^2} S^2\right] \\
&= \frac{\sigma^2}{n} E\left[\chi^2_{n-1}\right] \\
&= \frac{(n-1)\sigma^2}{n}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(\widehat{\sigma^2}\right) &= Var\left[\frac{1}{n}\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2\right] \\
&= Var\left[\frac{n-1}{n} S^2\right] \\
&= \frac{\sigma^4}{n^2} E\left[\frac{n-1}{\sigma^2} S^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma^4}{n^2} E\left[\chi^2_{n-1}\right] \\
&= \frac{2(n-1)\sigma^4}{n^2}.
\end{aligned}
$$

Note that we have used the fact $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Furthermore, $S^2 = (1/(n-1))\sum_{i=1}^{n}(X_i - \bar{X})^2$ denotes the sample variance.