MATH20802: STATISTICAL METHODS
SECOND SEMESTER
ANSWERS TO THE IN CLASS TEST

1. For a continuous random random variable X let $M_X(t) = E[\exp(tX)]$ denote its moment generating function (mgf).

   (i) if $X \sim Bin(n, p)$ then

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{n} \exp(tx) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (p\exp(t))^x (1-p)^{n-x} \\
&= \frac{(p\exp(t)+1-p)^n}{(p\exp(t)+1-p)^n} \sum_{x=0}^{n} \binom{n}{x} (p\exp(t))^x (1-p)^{n-x} \\
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \frac{(p\exp(t))^x (1-p)^{n-x}}{(p\exp(t)+1-p)^n} \\
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \frac{(p\exp(t))^x (1-p)^{n-x}}{(p\exp(t)+1-p)^x \, (p\exp(t)+1-p)^{n-x}} \\
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \left(\frac{p\exp(t)}{p\exp(t)+1-p} \right)^x \left(\frac{1-p}{p\exp(t)+1-p} \right)^{n-x} \\
&= (p\exp(t)+1-p)^n \qquad \text{since } \sum_{x=0}^{n} \binom{n}{z} q^x (1-q)^x = 1.
\end{aligned}
$$

   Note that $\sum_{x=0}^{n} \binom{n}{z} q^x (1-q)^x = 1$ because the PMF of a binomial distribution (in fact, the PMF of any discrete distribution) must add to 1.

   (ii) if X is a discrete uniform random variable with the pmf $p(x) = 1/N$ for $x = 1, 2, \ldots, N$ then

$$
\begin{aligned}
M_X(t) &= \frac{1}{N} \sum_{x=1}^{N} \exp(tx) \\
&= \frac{\exp(t)}{N} \sum_{x=1}^{N} \exp(t(x-1)) \\
&= \frac{\exp(t)\{1-\exp(Nt)\}}{N(1-\exp(t))}.
\end{aligned}
$$

   (iii) if $X \sim Uni(a, b)$ then

$$
\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_a^b \frac{\exp(tx)}{b-a} dx
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{b-a}\int_a^b \exp(tx)dx \\
&= \frac{1}{t(b-a)}\left[\exp(tx)\right]_a^b \\
&= \frac{\exp(bt)-\exp(at)}{t(b-a)}.
\end{aligned}$$

(iv) if $X \sim Ga(a,\lambda)$ then

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \frac{\lambda^a}{\Gamma(a)}\int_0^\infty x^{a-1}\exp(tx-\lambda x)dx \\
&= \frac{\lambda^a}{(\lambda-t)^a\Gamma(a)}\int_0^\infty y^{a-1}\exp(-y)dy \\
&= \frac{\lambda^a}{(\lambda-t)^a}
\end{aligned}$$

provided that $\lambda - t < 0$ (we have set $y = (\lambda - t)x$).

(v) if $X \sim N(\mu,\sigma^2)$ then

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_{-\infty}^\infty \exp(tx)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}+tx\right\}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left\{-\frac{x^2-2\mu x-2\sigma^2 tx+\mu^2}{2\sigma^2}\right\}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2+\mu^2-(\mu+\sigma^2 t)^2}{2\sigma^2}\right\}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left\{-\frac{(x-\mu-\sigma^2 t)^2-2\mu\sigma^2 t-\sigma^4 t^2}{2\sigma^2}\right\}dx \\
&= \exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left\{-\frac{(x-\mu-\sigma t)^2}{2\sigma^2}\right\}dx \\
&= \exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right).
\end{aligned}$$

2. Consider the two independent random samples: $X_1, X_2, \ldots, X_n$ from $N(\mu_X, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_m$ from $N(\mu_Y, \sigma^2)$, where σ^2 is assumed known. The parameters μ_X and μ_Y are assumed not known.

   (i) The likelihood function of μ_X and μ_Y is

$$\begin{aligned}
L(\mu_X,\mu_Y) &= \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(X_i-\mu_X)^2}{2\sigma^2}\right\}\right)\left(\prod_{i=1}^m \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(Y_i-\mu_Y)^2}{2\sigma^2}\right\}\right) \\
&= \frac{1}{(2\pi)^{(m+n)/2}\sigma^{m+n}}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (X_i-\mu_X)^2+\sum_{i=1}^m (Y_i-\mu_Y)^2\right]\right\}.
\end{aligned}$$

(ii) The log likelihood function of μ_X and μ_Y is

$$l(\mu_X, \mu_Y) = -\frac{m+n}{2}\log(2\pi) - (m+n)\log\sigma - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i - \mu_X)^2 + \sum_{i=1}^{m}(Y_i - \mu_Y)^2\right].$$

The partial derivatives with respect to μ_X and μ_Y are

$$\frac{\partial l(\mu_X, \mu_Y)}{\partial \mu_X} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu_X)$$

and

$$\frac{\partial l(\mu_X, \mu_Y)}{\partial \mu_Y} = \frac{1}{\sigma^2}\sum_{i=1}^{m}(Y_i - \mu_Y).$$

Setting $\partial l(\mu_X, \mu_Y)/\partial \mu_X = 0$, one obtains

$$\begin{aligned} & \sum_{i=1}^{n}(X_i - \mu_X) = 0 \\ \Rightarrow \quad & n\mu_X = \sum_{i=1}^{n} X_i \\ \Rightarrow \quad & \mu_X = \bar{X} \end{aligned}$$

where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$. Similarly, setting $\partial l(\mu_X, \mu_Y)/\partial \mu_Y = 0$, one obtains

$$\begin{aligned} & \sum_{i=1}^{m}(Y_i - \mu_Y) = 0 \\ \Rightarrow \quad & m\mu_Y = \sum_{i=1}^{m} Y_i \\ \Rightarrow \quad & \mu_Y = \bar{Y} \end{aligned}$$

where $\bar{Y} = (1/m)\sum_{i=1}^{m} Y_i$. So, the mles are $\widehat{\mu_X} = \bar{X}$ and $\widehat{\mu_Y} = \bar{Y}$.

(iii) Note $X - Y \sim N(\mu_X - \mu_Y, 2\sigma^2)$. So,

$$\begin{aligned} \Pr(X < Y) &= \Pr(X - Y < 0) \\ &= \Pr\left(\frac{X - Y - (\mu_X - \mu_Y)}{\sqrt{2}\sigma} < \frac{0 - (\mu_X - \mu_Y)}{\sqrt{2}\sigma}\right) \\ &= \Pr\left(Z < \frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right) \\ &= \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{2}\sigma}\right) \end{aligned}$$

and so the mle of $\Pr(X < Y)$ is $\Phi((\widehat{\mu_Y} - \widehat{\mu_X})/(\sqrt{2}\sigma))$.

(iv) Note $\bar{X} \sim N(\mu_X, \sigma^2/n)$ and so $E(\widehat{\mu_X}) = \mu_X$ and $Var(\widehat{\mu_X}) = \sigma^2/n$. So, $\widehat{\mu_X}$ is an unbiased and consistent estimator for μ_X.

(v) Note $\bar{Y} \sim N(\mu_Y, \sigma^2/m)$ and so $E(\widehat{\mu_Y}) = \mu_Y$ and $Var(\widehat{\mu_Y}) = \sigma^2/m$. So, $\widehat{\mu_Y}$ is an unbiased and consistent estimator for μ_Y.