# MATH20802: STATISTICAL METHODS
# SECOND SEMESTER
# ANSWERS TO THE IN CLASS TEST

## ANSWERS TO QUESTION 1

### Solutions to Question 1

(i) Setting $z = \exp\{-y/\beta\}$, we obtain the cumulative distribution function as

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{1}{\beta} \exp\left(-\frac{y}{\beta}\right) \exp\left\{-\exp\left(-\frac{y}{\beta}\right)\right\} dy \\
&= \int_{\exp\{-x/\beta\}}^{\infty} \frac{1}{\beta} z \exp\{-z\} \frac{\beta}{z} dz \\
&= \int_{\exp\{-x/\beta\}}^{\infty} \exp\{-z\} dz \\
&= \left[-\exp(-z)\right]_{\exp\{-x/\beta\}}^{\infty} \\
&= \exp\left\{-\exp\left(-\frac{x}{\beta}\right)\right\}.
\end{aligned}
$$

(4 marks)

(ii) Setting $z = \exp\{-x/\beta\}$, we obtain the moment generating function as

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} \frac{1}{\beta} \exp\left(tx - \frac{x}{\beta}\right) \exp\left\{-\exp\left(-\frac{x}{\beta}\right)\right\} dx \\
&= \int_{0}^{\infty} \frac{1}{\beta} \exp\{-t\beta \log z\} z \exp\{-z\} \frac{\beta}{z} dz \\
&= \int_{0}^{\infty} \exp\{-t\beta \log z\} \exp\{-z\} dz \\
&= \int_{0}^{\infty} z^{-\beta t} \exp\{-z\} dz \\
&= \Gamma(1 - \beta t),
\end{aligned}
$$

where the last step follows by the definition of the gamma function. (4 marks)

(iii) the first derivative of $M_X(t)$ is

$$M_X^{'}(t) = -\beta \Gamma^{'}(1 - \beta t).$$

So, $E(X) = M_X^{'}(0) = -\beta \Gamma^{'}(1)$. (2 marks)

**ANSWERS TO QUESTION 2**

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with the common probability mass function (pmf):

$$p(x) = \theta(1-\theta)^{x-1}$$

for $x = 1, 2, \ldots$ and $0 < \theta < 1$. This pmf corresponds to the geometric distribution, so $E(X_i) = 1/\theta$ and $Var(X_i) = (1-\theta)/\theta^2$.

(i) The likelihood function of $\theta$ is

$$L(\theta) = \prod_{i=1}^{n} \left\{ \theta(1-\theta)^{X_i - 1} \right\} = \theta^n (1-\theta)^{\sum_{i=1}^{n} X_i - n}.$$

(2 marks)

(ii) The log likelihood function of $\theta$ is

$$\log L(\theta) = n \log \theta + \left( \sum_{i=1}^{n} X_i - n \right) \log(1-\theta).$$

The first and second derivatives of this with respect to $\theta$ are

$$\frac{d \log L(\theta)}{d\theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} X_i - n}{1-\theta}$$

and

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} X_i - n}{(1-\theta)^2},$$

respectively. Note that $d \log L(\theta)/d\theta = 0$ if $\theta = n/\sum_{i=1}^{n} X_i$ and that $d^2 \log L(\theta)/d\theta^2 < 0$ for all $0 < \theta < 1$. So, it follows that $\widehat{\theta} = n/\sum_{i=1}^{n} X_i$ is the mle of $\theta$. (2 marks)

(iii) By the invariance principle, the mle of $\psi = 1/\theta$ is $\widehat{\psi} = (1/n)\sum_{i=1}^{n} X_i$. (2 marks)

(iv) The bias of $\widehat{\psi}$ is

$$\begin{aligned}
E\left(\widehat{\psi}\right) - \psi &= E\left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) - \psi \\
&= \frac{1}{n} \sum_{i=1}^{n} E(X_i) - \psi \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\theta} - \psi \\
&= \psi - \psi \\
&= 0.
\end{aligned}$$

The variance of $\widehat{\psi}$ is

$$\begin{aligned} Var\left(\widehat{\psi}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} \frac{1-\theta}{\theta^2} \\ &= \frac{1-\theta}{n\theta^2} \\ &= \frac{\psi^2-\psi}{n}. \end{aligned}$$

The mean squared error of $\widehat{\psi}$ is

$$MSE\left(\widehat{\psi}\right) = \frac{\psi^2-\psi}{n}.$$

(2 marks)

(v) The mle of $\psi$ is unbiased and consistent. (2 marks)