MATH20802: STATISTICAL METHODS
SOME RANDOM QUESTIONS WITH SOLUTIONS

2010/2011 in-class test, question 1(iii). if X is a continuous random variable with the pdf $f(x) = \exp(-x)/\{1+\exp(-x)\}^2$ for $-\infty < x < \infty$ then

$$M_X(t) = E\left[\exp(tX)\right] = \int_{-\infty}^{\infty} \frac{\exp(tx-x)}{\left[1+\exp(-x)\right]^2} dx. \tag{1}$$

Set $y = 1/\left\{1+\exp(-x)\right\}$, so

$$\begin{aligned} & 1+\exp(-x) = \frac{1}{y} \\ \Longrightarrow\ & \exp(-x) = \frac{1}{y} - 1 = \frac{1-y}{y} \\ \Longrightarrow\ & x = -\log\left(\frac{1-y}{y}\right) \\ \Longrightarrow\ & x = \log y - \log(1-y) \end{aligned}$$

and

$$\frac{dx}{dy} = \frac{d\log y}{dy} - \frac{d\log(1-y)}{dy} = \frac{1}{y} + \frac{1}{1-y} = \frac{1}{y(1-y)}.$$

Substituting these into (1) gives

$$\begin{aligned} M_X(t) &= \int_0^1 \left[\exp(-x)\right]^{1-t} \frac{1}{\left[1+\exp(-x)\right]^2} \frac{dx}{dy} dy \\ &= \int_0^1 \left[\frac{1-y}{y}\right]^{1-t} y^2 \frac{1}{y(1-y)} dy \\ &= \int_0^1 (1-y)^{-t} y^t dy \\ &= B(1-t, 1+t). \end{aligned}$$

2010/2011 in-class test, question 2(ii). The joint likelihood function of θ_1 and θ_2 is

$$\begin{aligned} L\left(\theta_1, \theta_2\right) &= \prod_{i=1}^n \left[\theta_2 x_i^{\theta_2 - 1} \theta_1^{-\theta_2} I\left\{0 < x_i < \theta_1\right\}\right] \\ &= \theta_2^n \left(\prod_{i=1}^n x_i\right)^{\theta_2 - 1} \theta_1^{-n\theta_2} \left(\prod_{i=1}^n I\left\{0 < x_i < \theta_1\right\}\right) \\ &= \theta_2^n \left(\prod_{i=1}^n x_i\right)^{\theta_2 - 1} \theta_1^{-n\theta_2} I\left\{\max\left(x_1, x_2, \ldots, x_n\right) < \theta_1\right\} \end{aligned}$$

for $\theta_1 > 0$ and $\theta_2 > 0$.

The distributions of max and min of IID random variables. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variable with common CDF F. Let $Y =$

$\min(X_1, X_2, \ldots, X_n)$. Then cdf CDF of Y is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \le y) \\
&= \Pr(\min(X_1, X_2, \ldots, X_n) \le y) \\
&= 1 - \Pr(\min(X_1, X_2, \ldots, X_n) > y) \\
&= 1 - \Pr(X_1 > y, X_2 > y, \ldots, X_n > y) \\
&= 1 - \Pr(X_1 > y)\Pr(X_2 > y)\cdots\Pr(X_n > y) \\
&\qquad [\text{because of independce of } X_1, X_2, \ldots, X_n] \\
&= 1 - [1 - \Pr(X_1 \le y)][1 - \Pr(X_2 \le y)]\cdots[1 - \Pr(X_n \le y)] \\
&= 1 - [1 - F(y)][1 - F(y)]\cdots[1 - F(y)] \\
&= 1 - [1 - F(y)]^n .
\end{aligned}$$

Now let $Z = \max(X_1, X_2, \ldots, X_n)$. Then cdf CDF of Z is

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) \\
&= \Pr(\max(X_1, X_2, \ldots, X_n) \le z) \\
&= \Pr(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\
&= \Pr(X_1 \le y)\Pr(X_2 \le y)\cdots\Pr(X_n \le y) \\
&\qquad [\text{because of independce of } X_1, X_2, \ldots, X_n] \\
&= F(y)\,F(y)\cdots F(y) \\
&= [F(y)]^n .
\end{aligned}$$

The first two moments of a chisquare random variable. Suppose X is a chisquare random variable with degree of freedom ν. Then its expectation can be derived as

$$\begin{aligned}
E(X) &= \int_0^\infty x \frac{x^{\frac{\nu}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx \\
&= \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} \frac{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx \\
&= \frac{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} dx \\
&= \frac{2\Gamma\left(\frac{\nu}{2}+1\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} dx \\
&= \frac{2\frac{\nu}{2}\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} dx \qquad \text{since } \Gamma(x+1) = x\Gamma(x) \\
&= \nu \int_0^\infty \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} dx \qquad \text{since } \frac{x^{\frac{\nu+2}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+2}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} \text{ is a chisquare pdf} \\
&= \nu.
\end{aligned}
$$

Similarly, $E\left(X^2\right)$ can be derived as

$$
\begin{aligned}
E\left(X^2\right) &= \int_0^\infty x^2 \frac{x^{\frac{\nu}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx \\
&= \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx \\
&= \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)} \frac{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} dx \\
&= \frac{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)} dx \\
&= \frac{4\Gamma\left(\frac{\nu}{2}+2\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+2}{2}\right)} dx \\
&= \frac{4\frac{\nu}{2}\left(\frac{\nu}{2}+1\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)} dx \qquad \text{since } \Gamma(x+2) = x(x+1)\Gamma(x) \\
&= \nu(\nu+2) \int_0^\infty \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)} dx \qquad \text{since } \frac{x^{\frac{\nu+4}{2}-1}\exp\left(-\frac{x}{2}\right)}{2^{\frac{\nu+4}{2}}\Gamma\left(\frac{\nu+4}{2}\right)} \text{ is a chisquare pdf} \\
&= \nu(\nu+2).
\end{aligned}
$$

So, Var $(X) = E\left(X^2\right) - (E(X))^2 = \nu(\nu+2) - \nu^2 = 2\nu.$

Solution to example 31 in the booklet. The likelihood function is

$$L(\delta) = \prod_{i=1}^{n} \left[e^{\delta - x_i} I\left\{ x_i \geq \delta \right\} \right] = e^{n\delta - \sum_{i=1}^{n} x_i} I\left\{ \min\left(x_1, \ldots, x_n\right) \geq \delta \right\}.$$

Graph this as a function of δ. Note that $e^{n\delta}$ is an increasing function of δ. But δ must be less than or equal to $\min\left(x_1, \ldots, x_n\right)$. So, the largest point of the graph will be attained when $\delta = \min\left(x_1, \ldots, x_n\right)$. Hence, $\min\left(x_1, \ldots, x_n\right)$ is the MLE of δ.

Solution to example 32 in the booklet. The likelihood function and its log are

$$L(\theta) = \prod_{i=1}^{n} \left[(\theta + 1)\, x_i^{\theta} \right] = (\theta + 1)^n \left(\prod_{i=1}^{n} x_i \right)^{\theta}$$

and

$$\log L(\theta) = n \log(\theta + 1) + \theta \sum_{i=1}^{n} \log x_i.$$

The derivative of $\log L$ with respect to θ is

$$\frac{d \log L(\theta)}{d\theta} = \frac{n}{\theta + 1} + \sum_{i=1}^{n} \log x_i.$$

Setting this to zero, we obtain the solution

$$\widehat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i} - 1.$$

This is an MLE since

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{n}{(\theta + 1)^2} < 0.$$

Solution to example 33 in the booklet. The likelihood function and its log are

$$L(p) = \prod_{i=1}^{n} \left[p\,(1 - p)^{x_i - 1} \right] = p^n (1 - p)^{\left(\sum_{i=1}^{n} x_i\right) - n}$$

and

$$\log L(p) = n \log p + \left[\left(\sum_{i=1}^{n} x_i \right) - n \right] \log(1 - p).$$

The derivative of $\log L$ with respect to p is

$$\frac{d \log L(p)}{dp} = \frac{n}{p} - \left[\left(\sum_{i=1}^{n} x_i \right) - n \right] \frac{1}{1 - p}.$$

Setting this to zero, we obtain the solution

$$\widehat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\overline{X}}.$$

This is an MLE since

$$\frac{d^2 \log L(p)}{dp^2} = -\frac{n}{p^2} - \left[\left(\sum_{i=1}^{n} x_i \right) - n \right] \frac{1}{(1-p)^2} < 0.$$

Note that $\left(\sum_{i=1}^{n} x_i\right) - n \geq 0$ since $x_i \geq 1$ for all i.

Solutions to example 14 in the booklet. First we derive the biases of the two estimators.

$$\begin{aligned}
\text{Bias}\left(\widehat{p_1}\right) &= E\left(\widehat{p_1}\right) - p \\
&= E\left[\frac{1}{2}\left(\frac{X}{m} + \frac{Y}{n}\right)\right] - p \\
&= \frac{1}{2}\left[\frac{E(X)}{m} + \frac{E(Y)}{n}\right] - p \\
&= \frac{1}{2}\left(\frac{mp}{m} + \frac{np}{n}\right) - p \\
&= \frac{1}{2}(p + p) - p \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{Bias}\left(\widehat{p_2}\right) &= E\left(\widehat{p_2}\right) - p \\
&= E\left(\frac{X+Y}{m+n}\right) - p \\
&= \frac{E(X+Y)}{m+n} - p \\
&= \frac{E(X)+E(Y)}{m+n} - p \\
&= \frac{mp+np}{m+n} - p \\
&= p - p \\
&= 0.
\end{aligned}$$

So, both estimators are unbiased for p.

Now we derive the MSE of the two estimators.

$$\begin{aligned}
\text{MSE}\left(\widehat{p_1}\right) &= \text{Var}\left(\widehat{p_1}\right) \\
&= \text{Var}\left(\frac{1}{2}\left(\frac{X}{m} + \frac{Y}{n}\right)\right) \\
&= \frac{1}{4}\text{Var}\left(\frac{X}{m} + \frac{Y}{n}\right) \\
&= \frac{1}{4}\left[\frac{\text{Var}(X)}{m^2} + \frac{\text{Var}(Y)}{n^2}\right] \\
&= \frac{1}{4}\left[\frac{mp(1-p)}{m^2} + \frac{np(1-p)}{n^2}\right] \\
&= \frac{1}{4}\left[\frac{p(1-p)}{m} + \frac{p(1-p)}{n}\right] \\
&= \frac{p(1-p)}{4}\left(\frac{1}{m} + \frac{1}{n}\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}\left(\widehat{p_2}\right) &= \text{Var}\left(\widehat{p_2}\right) \\
&= \text{Var}\left(\frac{X+Y}{m+n}\right) \\
&= \frac{1}{(m+n)^2}\text{Var}\left(X+Y\right) \\
&= \frac{1}{(m+n)^2}\left[\text{Var}(X)+\text{Var}(Y)\right] \\
&= \frac{1}{(m+n)^2}\left[mp(1-p)+np(1-p)\right] \\
&= \frac{p(1-p)}{m+n}.
\end{aligned}$$

$\widehat{p_2}$ has smaller MSE than $\widehat{p_1}$ since

$$\begin{aligned}
& \frac{p(1-p)}{m+n} \leq \frac{p(1-p)}{4}\left(\frac{1}{m}+\frac{1}{n}\right) \\
\Longleftrightarrow\; & \frac{1}{m+n} \leq \frac{1}{4}\left(\frac{1}{m}+\frac{1}{n}\right) \\
\Longleftrightarrow\; & \frac{1}{m+n} \leq \frac{1}{4}\frac{m+n}{mn} \\
\Longleftrightarrow\; & 4mn \leq (m+n)^2 \\
\Longleftrightarrow\; & 4mn \leq m^2+n^2+2mn \\
\Longleftrightarrow\; & 0 \leq m^2+n^2-2mn \\
\Longleftrightarrow\; & 0 \leq (m-n)^2.
\end{aligned}$$

The first two moments of a negative binomial random variable. Suppose X is a negative binomial random variable with pmf given by equation (9) in the lecture notes. Then its expectation can be derived as

$$\begin{aligned}
E(X) &= \sum_{x=r}^{\infty} x\binom{x-1}{r-1}p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty}(x-r+r)\binom{x-1}{r-1}p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty}(x-r)\binom{x-1}{r-1}p^r(1-p)^{x-r} + r\sum_{x=r}^{\infty}\binom{x-1}{r-1}p^r(1-p)^{x-r} \\
&= \left[\sum_{x=r+1}^{\infty}(x-r)\binom{x-1}{r-1}p^r(1-p)^{x-r}\right]+r \\
&\quad \text{since } \binom{x-1}{r-1}p^r(1-p)^{x-r} \text{ is a negative binomial pmf with parameters } r \text{ and } p \\
&= \left[\sum_{x=r+1}^{\infty}(x-r)\frac{(x-1)!}{(r-1)!(x-r)!}p^r(1-p)^{x-r}\right]+r
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{x=r+1}^{\infty} \frac{(x-1)!}{(r-1)!(x-r-1)!} p^r (1-p)^{x-r} \right] + r \\
&= \left[\sum_{x=r+1}^{\infty} \frac{r(x-1)!}{r(r-1)!(x-r-1)!} p^{r+1-1} (1-p)^{x-r-1+1} \right] + r \\
&= \left[\frac{r(1-p)}{p} \sum_{x=r+1}^{\infty} \frac{(x-1)!}{r!(x-r-1)!} p^{r+1} (1-p)^{x-r-1} \right] + r \\
&= \left[\frac{r(1-p)}{p} \sum_{x=r+1}^{\infty} \binom{x-1}{r} p^{r+1} (1-p)^{x-r-1} \right] + r \\
&= \left[\frac{r(1-p)}{p} \right] + r \\
& \quad \text{since } \binom{x-1}{r} p^{r+1} (1-p)^{x-r-1} \text{ is a negative binomial pmf with parameters } r+1 \text{ and } p \\
&= \frac{r}{p}.
\end{aligned}
$$

$E\left(X^2\right)$ can be derived similarly by writing

$$
\begin{aligned}
E\left(X^2\right) &= \sum_{x=r}^{\infty} x^2 \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} (x-r+r)^2 \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \left[(x-r)^2 + 2(x-r) + r^2 \right] \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} (x-r)^2 \binom{x-1}{r-1} p^r (1-p)^{x-r} + 2 \sum_{x=r}^{\infty} (x-r) \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
& \quad + r^2 \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r}
\end{aligned}
$$

Solutions to question 1, 2013/2014 in-class test. The following solution is correct if X has the pdf

$$
f_X(x) = \frac{x^{a/2} \exp(-x/2)}{2^{a/2} \Gamma(a/2)}
$$

for $x > 0$ and $a > 0$.

(i) The moment generating function of X is

$$
M_X(t) = E\left[\exp(tX)\right]
$$

$$
\begin{aligned}
&= \int_0^\infty \exp(tx) \frac{x^{a/2} \exp(-x/2)}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{x^{a/2} \exp\left[-\left(1/2 - t\right) x\right]}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{y^{a/2} \exp\left(-y\right)}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} dx \qquad \text{substituting } y = \left(1/2 - t\right) x \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} \int_0^\infty y^{a/2} \exp\left(-y\right) dx \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} \Gamma(a/2 + 1) \qquad \text{by definition of gamma function} \\
&= \frac{1}{2^{a/2}\Gamma(a/2) \left(1/2 - t\right)^{a/2+1}} (a/2)\Gamma(a/2) \qquad \text{by using the fact } \Gamma(s+1) = s\Gamma(s) \\
&= a \left(1 - 2t\right)^{-a/2-1}.
\end{aligned}
$$

(ii) We have $E(Y) = M_X(1)$ since $E(Y) = E(\exp(X))$. We have $E(Y^2) = M_X(2)$ since $E(Y^2) = E(\exp(2X))$.

(iii) From (ii), we have

$$E(Y) = M_X(1) = a(-1)^{-a/2-1}$$

and

$$E(Y^2) = M_X(2) = a(-3)^{-a/2-1}.$$

So,

$$Var(Y) = a(-3)^{-a/2-1} - a^2(-1)^{-a-2}.$$

(iv) We have

$$
\begin{aligned}
M_{X_1+X_2}(t) &= E\left[\exp\left(tX_1 + tX_2\right)\right] \\
&= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\
&= a_1 a_2 \left(1 - 2t\right)^{-a_1/2 - a_2/2 - 2}.
\end{aligned}
$$

(v) It follows from (iv) that $X_1 + X_2$ is a gamma random variable with $\lambda = 1/2$ and $a = \left(a_1 + a_2\right)/2 + 2$ if $a_1 a_2 = 1$.

The following solution is correct if X has the pdf

$$f_X(x) = \frac{x^{a/2-1} \exp(-x/2)}{2^{a/2}\Gamma(a/2)}$$

for $x > 0$ and $a > 0$.

(i) The moment generating function of X is

$$\begin{aligned}
M_X(t) &= E\left[\exp(tX)\right] \\
&= \int_0^\infty \exp(tx) \frac{x^{a/2-1}\exp(-x/2)}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{x^{a/2-1}\exp\left[-\left(1/2-t\right)x\right]}{2^{a/2}\Gamma(a/2)} dx \\
&= \int_0^\infty \frac{y^{a/2-1}\exp\left(-y\right)}{2^{a/2}\Gamma(a/2)\left(1/2-t\right)^{a/2}} dx \\
&= \frac{1}{2^{a/2}\Gamma(a/2)\left(1/2-t\right)^{a/2}} \int_0^\infty y^{a/2-1}\exp\left(-y\right) dx \\
&= \frac{1}{2^{a/2}\Gamma(a/2)\left(1/2-t\right)^{a/2}} \Gamma(a/2) \\
&= \frac{1}{2^{a/2}\left(1/2-t\right)^{a/2}} \\
&= (1-2t)^{-a/2}.
\end{aligned}$$

(ii) We have $E(Y) = M_X(1)$ since $E(Y) = E(\exp(X))$. We have $E(Y^2) = M_X(2)$ since $E(Y^2) = E(\exp(2X))$.

(iii) From (ii), we have

$$E(Y) = M_X(1) = (-1)^{a/2}$$

and

$$E(Y^2) = M_X(2) = (-3)^{a/2}.$$

So,

$$Var(Y) = (-3)^{a/2} - (-1)^a.$$

(iv) We have

$$\begin{aligned}
M_{X_1+X_2}(t) &= E\left[\exp\left(tX_1 + tX_2\right)\right] \\
&= E\left[\exp(tX_1)\right] E\left[\exp(tX_2)\right] \\
&= (1-2t)^{-a_1/2} (1-2t)^{-a_2/2} \\
&= (1-2t)^{-(a_1+a_2)/2}.
\end{aligned}$$

(v) It follows from (iv) that $X_1 + X_2$ is a chisquare random variable with degree of freedom equal to $a_1 + a_2$.

Solutions to question 2(i), 2013/2014 in-class test. The joint likelihood function of μ and σ^2 is

$$\begin{aligned}
L\left(\mu, \sigma^2\right) &= \prod_{i=1}^n \left\{ \frac{1}{\sqrt{2\pi}\sqrt{c}\sigma} \exp\left[-\frac{\left(X_i-\mu\right)^2}{2c\sigma^2}\right] \right\} \\
&= \frac{1}{(2\pi)^{n/2} c^{n/2} \sigma^n} \exp\left[-\frac{1}{2c\sigma^2} \sum_{i=1}^n \left(X_i-\mu\right)^2\right].
\end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log c - n\log\sigma - \frac{1}{2c\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2.$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{c\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right) = \frac{1}{c\sigma^2}\left[\left(\sum_{i=1}^{n} X_i\right) - n\mu\right]$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{c\sigma^3}\sum_{i=1}^{n}\left(X_i - \mu\right)^2,$$

respectively.

Solutions to question 2(i), 2014/2015 in-class test. (i) The joint likelihood function of μ and σ^2 is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^{n}\left\{\frac{1}{\sqrt{2\pi}\sqrt{c}\sigma}\exp\left[-\frac{\left(X_i - c\mu\right)^2}{2c\sigma^2}\right]\right\} \\ &= \frac{1}{(2\pi)^{n/2}c^{n/2}\sigma^n}\exp\left[-\frac{1}{2c\sigma^2}\sum_{i=1}^{n}\left(X_i - c\mu\right)^2\right]. \end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log c - n\log\sigma - \frac{1}{2c\sigma^2}\sum_{i=1}^{n}\left(X_i - c\mu\right)^2.$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - c\mu\right) = \frac{1}{\sigma^2}\left[\left(\sum_{i=1}^{n} X_i\right) - nc\mu\right]$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{c\sigma^3}\sum_{i=1}^{n}\left(X_i - c\mu\right)^2,$$

respectively.

Solutions to question 1, part (i), 2007/2008 in-class test. if $X \sim Bin(n, p)$ then

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{n}\exp(tx)\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}(p\exp(t))^x(1-p)^{n-x} \\ &= \frac{(p\exp(t)+1-p)^n}{(p\exp(t)+1-p)^n}\sum_{x=0}^{n}\binom{n}{x}(p\exp(t))^x(1-p)^{n-x} \end{aligned}$$

$$
\begin{aligned}
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \frac{(p\exp(t))^x (1-p)^{n-x}}{(p\exp(t)+1-p)^n} \\
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \frac{(p\exp(t))^x (1-p)^{n-x}}{(p\exp(t)+1-p)^x\,(p\exp(t)+1-p)^{n-x}} \\
&= (p\exp(t)+1-p)^n \sum_{x=0}^{n} \binom{n}{x} \left(\frac{p\exp(t)}{p\exp(t)+1-p}\right)^x \left(\frac{1-p}{p\exp(t)+1-p}\right)^{n-x} \\
&= (p\exp(t)+1-p)^n \qquad \text{since } \sum_{x=0}^{n} \binom{n}{z} q^x (1-q)^x = 1.
\end{aligned}
$$

Note that $\sum_{x=0}^{n} \binom{n}{z} q^x (1-q)^x = 1$ because the PMF of a binomial distribution (in fact, the PMF of any discrete distribution) must add to 1.

2011/2012 in-class test, question 1, part (iii). Note that

$$
Var\left(\frac{X_1+\overline{X}}{2}\right) = \frac{1}{4} Var\left(X_1+\overline{X}\right) \neq \frac{1}{4}\left[Var\left(X_1\right) + Var\left(\overline{X}\right)\right]
$$

since X_1 and $\overline{X}$ are not independent. But we can write

$$
\begin{aligned}
Var\left(\frac{X_1+\overline{X}}{2}\right) &= \frac{1}{4} Var\left(X_1+\overline{X}\right) \\
&= \frac{1}{4} Var\left(X_1+\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{4} Var\left(X_1+\frac{X_1}{n}+\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right) \\
&= \frac{1}{4} Var\left(\left(1+\frac{1}{n}\right)X_1+\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right) \\
&= \frac{1}{4}\left[Var\left(\left(1+\frac{1}{n}\right)X_1\right) + Var\left(\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right)\right] \\
&\qquad \text{since } X_1 \text{ and } X_2,\ldots,X_n \text{ are independent} \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 Var\left(X_1\right) + \frac{1}{n^2} Var\left(X_2+\cdots+X_n\right)\right] \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 Var\left(X_1\right) + \frac{1}{n^2} Var\left(\sum_{i=2}^{n} X_i\right)\right] \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 Var\left(X_1\right) + \frac{1}{n^2} \sum_{i=2}^{n} Var\left(X_i\right)\right] \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 \sigma^2 + \frac{1}{n^2} \sum_{i=2}^{n} \sigma^2\right]
\end{aligned}
$$

$$= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2\sigma^2 + \frac{1}{n^2}(n-1)\sigma^2\right].$$

The limit of this as $n \to \infty$ is $\sigma^2/4$, so the estimator is not consistent.

Solution to question 2(iv), 2013/14 in-class test. The mle, $\widehat{\mu}$, is an unbiased and consistent estimator for μ since

$$\begin{aligned} E(\widehat{\mu}) &= E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^n E(X_i) \\ &= \frac{1}{n}\sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var(\widehat{\mu}) &= Var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^n c\sigma^2 \\ &= \frac{c\sigma^2}{n}. \end{aligned}$$

2011/2012 in-class test, question 1, part (i). Note that

$$\begin{aligned} E(T(X)) &= 1\cdot\theta^3 + 2(1-c)\cdot\theta^2(1-\theta) + 2c\cdot\theta(1-\theta) + (1-2c)\cdot\theta(1-\theta)^2 \\ &= \theta^3 + 2(1-c)\cdot\left(\theta^2-\theta^3\right) + 2c\cdot\left(\theta-\theta^2\right) + (1-2c)\cdot\left(\theta-2\theta^2+\theta^3\right) \\ &= \theta^3 + 2(1-c)\cdot\theta^2 - 2(1-c)\cdot\theta^3 + 2c\cdot\theta - 2c\cdot\theta^2 + (1-2c)\cdot\theta - 2(1-2c)\cdot\theta^2 + (1-2c)\cdot\theta^3 \\ &= (2c+1-2c)\cdot\theta + (2-2c-2c-2+4c)\cdot\theta^2 + (1-2+2c+1-2c)\cdot\theta^3 \\ &= 1\cdot\theta + 0\cdot\theta^2 + 0\cdot\theta^3 \\ &= \theta \end{aligned}$$

Hence, $T(X)$ is unbiased for all c.

2012/2013 in-class test, question 1, part (ii). Note that

$$Var\left(\frac{X_1+\overline{X}}{2}\right) = \frac{1}{4}Var\left(X_1+\overline{X}\right) \neq \frac{1}{4}\left[Var(X_1) + Var\left(\overline{X}\right)\right]$$

since X_1 and $\overline{X}$ are not independent. But we can write

$$\begin{aligned}
Var\left(\frac{X_1+\overline{X}}{2}\right) &= \frac{1}{4}Var\left(X_1+\overline{X}\right) \\
&= \frac{1}{4}Var\left(X_1+\frac{1}{n}\sum_{i=1}^{n}X_i\right) \\
&= \frac{1}{4}Var\left(X_1+\frac{X_1}{n}+\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right) \\
&= \frac{1}{4}Var\left(\left(1+\frac{1}{n}\right)X_1+\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right) \\
&= \frac{1}{4}\left[Var\left(\left(1+\frac{1}{n}\right)X_1\right)+Var\left(\frac{X_2}{n}+\cdots+\frac{X_n}{n}\right)\right] \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 Var\left(X_1\right)+\frac{1}{n^2}Var\left(X_2+\cdots+X_n\right)\right] \\
&= \frac{1}{4}\left[\left(1+\frac{1}{n}\right)^2 Var\left(X_1\right)+\frac{1}{n^2}Var\left(\sum_{i=2}^{n}X_i\right)\right]
\end{aligned}$$

since X_1 and $X_2, \ldots, X_n$ are independent.

problem 6, sheet 7. I defined $F_{\nu_1,\nu_2,\alpha}$ as

$$\Pr\left(F_{\nu_1,\nu_2} < F_{\nu_1,\nu_2,\alpha}\right) = 1-\alpha, \tag{2}$$

this definition may be different from those you saw in other courses. It follows from (1) that

$$\begin{aligned}
& \Pr\left(\frac{1}{F_{\nu_1,\nu_2}} > \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = 1-\alpha \\
\Longrightarrow\quad & 1-\Pr\left(\frac{1}{F_{\nu_1,\nu_2}} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = 1-\alpha \\
\Longrightarrow\quad & \Pr\left(\frac{1}{F_{\nu_1,\nu_2}} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = \alpha \\
\Longrightarrow\quad & \Pr\left(F_{\nu_2,\nu_1} \leq \frac{1}{F_{\nu_1,\nu_2,\alpha}}\right) = \alpha \qquad \text{since } F_{\nu_2,\nu_1} = \tfrac{1}{F_{\nu_1,\nu_2}} \\
\Longrightarrow\quad & F_{\nu_2,\nu_1,1-\alpha} = \frac{1}{F_{\nu_1,\nu_2,\alpha}} \qquad \text{by definition of } F_{\nu_2,\nu_1,1-\alpha}
\end{aligned}$$

Solution to example 29 in the booklet. The likelihood function and its log are

$$L(\lambda) = \prod_{i=1}^{n}\left[\lambda\exp\left(-\lambda x_i\right)\right] = \lambda^n \exp\left(-\lambda\sum_{i=1}^{n}x_i\right)$$

and

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n}x_i.$$

The derivative of $\log L$ with respect to λ is

$$\frac{d\log L(\lambda)}{d\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i.$$

Setting this to zero, we obtain the solution

$$\widehat{\lambda}=\frac{n}{\sum_{i=1}^{n}x_i}=\frac{1}{\overline{X}}.$$

This is an MLE since

$$\frac{d^2\log L(\lambda)}{d\lambda^2}=-\frac{n}{\lambda^2}<0.$$

Note that

$$\Pr(X<1)=F(1)=1-\exp(-\lambda)\qquad\text{by equation (30) in lecture notes}$$

and

$$\text{mean}=E(X)=\frac{1}{\lambda}\qquad\text{by equation (32) in lecture notes.}$$

Hence their MLEs are

$$\Pr(\widehat{X<1})=1-\exp\left(-\widehat{\lambda}\right)=1-\exp\left(-\frac{1}{\overline{X}}\right)$$

and

$$\widehat{\text{mean}}=\frac{1}{\widehat{\lambda}}=\overline{X}.$$

Solutions to example 44 in the booklet. The likelihood function is

$$L\left(\beta_1\right)=\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\epsilon_i^2}{2\sigma^2}\right)\right]=\frac{1}{(2\pi)^{n/2}\sigma^n}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\epsilon_i^2\right)=\frac{1}{(2\pi)^{n/2}\sigma^n}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_1x_i\right)^2\right].$$

Its log is

$$\log L\left(\beta_1\right)=-\frac{n}{2}\log(2\pi)-n\log\sigma-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_1x_i\right)^2.$$

The derivative with respect to β_1 is

$$\frac{d\log L\left(\beta_1\right)}{d\beta_1}=\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i\left(y_i-\beta_1x_i\right).$$

Setting this to zero, we obtain

$$\begin{aligned}
&\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i\left(y_i-\beta_1x_i\right)=0\\
\Longrightarrow\ &\sum_{i=1}^{n}x_i\left(y_i-\beta_1x_i\right)=0\\
\Longrightarrow\ &\left(\sum_{i=1}^{n}x_iy_i\right)-\beta_1\left(\sum_{i=1}^{n}x_i^2\right)=0\\
\Longrightarrow\ &\widehat{\beta_1}=\left(\sum_{i=1}^{n}x_iy_i\right)/\left(\sum_{i=1}^{n}x_i^2\right).
\end{aligned}$$

This solution corresponds to an MLE since

$$\frac{d^2 \log L(\beta_1)}{d\beta_1^2} = \frac{d \log L(\beta_1)}{d\beta_1} \left[\frac{1}{\sigma^2} \sum_{i=1}^{n} x_i (y_i - \beta_1 x_i) \right] = -\frac{1}{\sigma^2} \sum_{i=1}^{n} x_i^2 < 0.$$

Solution to example 48 in the booklet. First find the mean and variance of the distribution as follows:

$$\mu = E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{6}{4} = \frac{3}{2} = 1.5$$

and

$$\sigma^2 = Var(X) = E\left(X^2\right) - [E(X)]^2 = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} - \left(\frac{3}{2}\right)^2 = \frac{14}{4} - \frac{9}{4} = \frac{5}{4} = 1.25.$$

Then

$$\begin{aligned}
& \Pr(1.4 < \overline{X} < 1.8) \\
= & \Pr\left(\frac{1.4 - \mu}{\sigma/\sqrt{n}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{1.8 - \mu}{\sigma/\sqrt{n}}\right) \\
= & \Pr\left(\frac{1.4 - 1.5}{\sqrt{1.25}/\sqrt{36}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{1.8 - 1.5}{\sqrt{1.25}/\sqrt{36}}\right) \\
= & \Pr\left(-\frac{0.1}{\sqrt{1.25}/6} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{0.3}{\sqrt{1.25}/6}\right) \\
= & \Pr\left(-\frac{0.6}{\sqrt{1.25}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{1.8}{\sqrt{1.25}}\right) \\
\approx & \Pr\left(-\frac{0.6}{\sqrt{1.25}} < N(0,1) < \frac{1.8}{\sqrt{1.25}}\right) \quad \text{by the Central Limit Theorem} \\
= & \Pr\left(N(0,1) < \frac{1.8}{\sqrt{1.25}}\right) - \Pr\left(N(0,1) < -\frac{0.6}{\sqrt{1.25}}\right) \\
= & \Phi\left(\frac{1.8}{\sqrt{1.25}}\right) - \Phi\left(-\frac{0.6}{\sqrt{1.25}}\right) \\
= & \Phi\left(\frac{1.8}{\sqrt{1.25}}\right) - \left[1 - \Phi\left(\frac{0.6}{\sqrt{1.25}}\right)\right] \\
= & \Phi\left(\frac{1.8}{\sqrt{1.25}}\right) - 1 + \Phi\left(\frac{0.6}{\sqrt{1.25}}\right) \\
= & 0.946 - 1 + 0.704 \quad \text{read from the normal tables} \\
= & 0.651.
\end{aligned}$$

Proof of $t_\nu \to N(0,1)$ as $\nu \to \infty$. Using the Stirling's formula

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z$$

as $z \to \infty$, we can write

$$\begin{aligned}
& \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \\
\sim\ & \frac{\sqrt{\frac{2\pi}{\frac{\nu+1}{2}}}\left(\frac{\nu+1}{2e}\right)^{\frac{\nu+1}{2}}}{\sqrt{\nu\pi}\sqrt{\frac{2\pi}{\frac{\nu}{2}}}\left(\frac{\nu}{2e}\right)^{\frac{\nu}{2}}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \\
\sim\ & \frac{1}{\sqrt{\pi}}\left(\frac{\nu+1}{\nu}\right)^{\frac{\nu}{2}}(2e)^{-\frac{1}{2}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}} \\
\sim\ & \frac{1}{\sqrt{\pi}}\sqrt{\left(1+\frac{1}{\nu}\right)^{\nu}}(2e)^{-\frac{1}{2}}\left[\left(1+\frac{x^2}{\nu}\right)^{\nu}\right]^{-\frac{1+\nu}{2\nu}} \\
\sim\ & \frac{1}{\sqrt{\pi}}\sqrt{e}(2e)^{-\frac{1}{2}}\left[\exp\left(x^2\right)\right]^{-\frac{1}{2}} \\
=\ & \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)
\end{aligned}$$

as $\nu \to \infty$. So, $t_\nu \to N(0,1)$ as $\nu \to \infty$.