Ex 1 Show that

$$C(u_1, u_2) = [u_1^{-\alpha} + u_2^{-\alpha} - 1]^{-\frac{1}{\alpha}}, \alpha > 0$$

is a copula.

(i) $C(u_1, 0) = [u_1^{-\alpha} + 0^{-\alpha} - 1]^{-\frac{1}{\alpha}}$

$= (\infty)^{-\frac{1}{\alpha}}$

$= 0$ ✓

(ii) $C(0, u_2) = [0^{-\alpha} + u_2^{-\alpha} - 1]^{-\frac{1}{\alpha}}$

$= (\infty)^{-\frac{1}{\alpha}}$

$= 0$ ✓

(iii) $C(u_1, 1) = [u_1^{-\alpha} + 1 - 1]^{-\frac{1}{\alpha}} = u_1$ ✓

(iv) $C(1, u_2) = [1 + u_2^{-\alpha} - 1]^{-\frac{1}{\alpha}} = u_2$ ✓

(v) $\frac{\partial}{\partial u_1} C(u_1, u_2) = \left(-\frac{1}{\alpha}\right) [u_1^{-\alpha} + u_2^{-\alpha} - 1]^{-\frac{1}{\alpha}-1}$

$\times (-\alpha) u_1^{-\alpha-1}$

$= [u_1^{-\alpha} + u_2^{-\alpha} - 1]^{-\frac{1}{\alpha}-1} u_1^{-\alpha-1}$

$\geqslant 0$ ✓

(vi) $\frac{\partial}{\partial u_2} C(u_1, u_2) = \left(-\frac{1}{\alpha}\right)\left[u_1^{-\alpha} + u_2^{-\alpha} - 1\right]^{-\frac{1}{\alpha}-1} (-\alpha) u_2^{-\alpha-1}$

$= \left[u_1^{-\alpha} + u_2^{-\alpha} - 1\right]^{-\frac{1}{\alpha}-1} \quad u_2^{-\alpha-1}$

≥ 0 ✓

Hence, C is a copula.

Ex 2 Show that

$$C_1(u_1, u_2) = [\min(u_1, u_2)]^{\theta} (u_1 u_2)^{1-\theta}, \quad 0 \leq \theta \leq 1$$

is a copula.

(i) $C_1(u_1, 0) = [\min(u_1, 0)]^{\theta} (u_1 \times 0)^{1-\theta}$
$= 0$ ✓

(ii) $C_1(0, u_2) = [\min(0, u_2)]^{\theta} (0 \times u_2)^{1-\theta}$
$= 0$ ✓

(iii) $C_1(u_1, 1) = [\min(u_1, 1)]^{\theta} u_1^{1-\theta} = u_1$ ✓

(iv) $C_1(1, u_2) = [\min(1, u_2)]^{\theta} u_2^{1-\theta} = u_2$ ✓

(v) $$\frac{\partial}{\partial u_1} C_1(u_1, u_2) = \frac{\partial}{\partial u_1} \begin{cases} u_1 u_2^{1-\theta} & u_1 \leq u_2 \\ u_1^{1-\theta} u_2 & u_1 > u_2 \end{cases}$$

$$= \begin{cases} u_2^{1-\theta} & u_1 \leq u_2 \\ (1-\theta) u_1^{-\theta} u_2 & u_1 > u_2 \end{cases}$$

≥ 0 ✓

(vi) $\frac{\partial}{\partial u_2} C'(u_1, u_2)$

$$= \frac{\partial}{\partial u_2} \begin{cases} u_1 u_2^{1-\theta} & u_1 \leq u_2 \\ u_1^{1-\theta} u_2 & u_1 > u_2 \end{cases}$$

$$= \begin{cases} (1-\theta) u_1 u_2^{-\theta} & u_1 \leq u_2 \\ u_1^{1-\theta} & u_1 > u_2 \end{cases}$$

≥ 0

Hence, C' is a copula.

Ex 3 Show that

$$C_1(u_1, u_2) = \frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)}, -1 \leq \theta < 1$$

is a copula.

(i) $C_1(u_1, 0) = \dfrac{u_1 \times 0}{1-\theta(1-u_1)(1-0)} = 0$ ✓

(ii) $C_1(0, u_2) = \dfrac{0 \times u_2}{1-\theta(1-0)(1-u_2)} = 0$ ✓

(iii) $C_1(u_1, 1) = \dfrac{u_1 \times 1}{1-\theta(1-u_1)(1-1)} = u_1$ ✓

(iv) $C_1(1, u_2) = \dfrac{1 \times u_2}{1-\theta(1-1)(1-u_2)} = u_2$ ✓

(v) $\dfrac{\partial}{\partial u_1} C_1(u_1, u_2) =$

$$\frac{u_2[1-\theta(1-u_1)(1-u_2)] - \theta u_1 u_2(1-u_2)}{[1-\theta(1-u_1)(1-u_2)]^2}$$

$$= \frac{u_2 [1-\theta(1-u_2)] \geq 0}{[1-\theta(1-u_1)(1-u_2)]^2} \geq 0 \checkmark$$

(vi) similar to (v).

Hence, C_1 is a copula.