Example I

Suppose $x_1, \ldots, x_n$ are IID $N(\mu, \sigma^2)$. Then

$$\widehat{ES}_p(X) = \bar{x} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \times \int_0^p \Phi^{-1}(t)\,dt$$

Find Bias & MSE of $\widehat{ES}_p(X)$.

$$\text{Bias}\left[\widehat{ES}_p(X)\right] = E\left[\widehat{ES}_p(X)\right] - ES_p(X)$$

$$= E(\bar{x}) + \frac{1}{p}\frac{1}{\sqrt{n}} E\left[\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] \times \int_0^p \Phi^{-1}(t)\,dt - \left[\mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt\right]$$

$$= E(\bar{x}) - \mu + \left\{\frac{1}{p\sqrt{n}} E\left[\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] - \frac{\sigma}{p}\right\}\int_0^p \Phi^{-1}(t)\,dt$$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \mu + \left\{\frac{1}{p\sqrt{n}} E\left[\sqrt{\sigma^2 \chi^2_{n-1}}\right] - \frac{\sigma}{p}\right\}\int_0^p \Phi^{-1}(t)\,dt$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i) - \mu + \frac{\sigma}{p}\left\{\frac{1}{\sqrt{n}} E\left[\sqrt{\chi^2_{n-1}}\right] - 1\right\} \times \int_0^p \Phi^{-1}(t)\,dt$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} \mu\right) - \mu + \frac{\sigma}{p}\left\{\frac{1}{\sqrt{n}} \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1\right\} \times \int_0^p \Phi^{-1}(t)\,dt$$

$$= \mu - \mu + \frac{\sigma}{p}\left\{\frac{1}{\sqrt{n}} \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1\right\} \int_0^p \Phi^{-1}(t)\,dt$$

$$= \frac{\sigma}{p}\left\{\frac{1}{\sqrt{n}} \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1\right\} \int_0^p \Phi^{-1}(t)\,dt$$

$\longrightarrow 0$ as $n \longrightarrow \infty$.

$\Rightarrow \widehat{ES}_p(X)$ is asymptotically unbiased.

$$MSE\left[\widehat{ES}_p(X)\right]$$

$$= Var\left[\widehat{ES}_p(X)\right] + \left[Bias(\widehat{ES}_p(X))\right]^2$$

$$Var\left[\widehat{ES}_p(X)\right]$$

$$= Var\left[\bar{x} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}\int_0^p \Phi^{-1}(t)\,dt\right]$$

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$$= Var(\bar{x}) + \frac{1}{p^2}\frac{1}{n} Var\left(\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right) \times \left[\int_0^p \Phi^{-1}(t)\,dt\right]^2$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{p^2 n} Var\left(\sqrt{\sigma^2 \chi^2_{n-1}}\right) \times \left[\int_0^p \Phi^{-1}(t)\,dt\right]^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) + \frac{\sigma^2}{p^2 n} Var\left(\sqrt{\chi^2_{n-1}}\right) \times \left[\int_0^p \Phi^{-1}(t)\,dt\right]^2$$

$$= \frac{1}{n^2} \times n\sigma^2 + \frac{\sigma^2}{p^2 n}\left\{n-1-2\left[\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right]^2\right\} \times \left[\int_0^p \Phi^{-1}(t)\,dt\right]^2$$

$$= \frac{\sigma^2}{n} \left[1 + \frac{1}{p^2} \left\{ n - 1 - 2 \left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right]^2 \right\} \times \left[\int_0^p \Phi^{-1}(t)\, dt \right]^2 \right]$$

$$\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Hence, $MSE\left[\widehat{ES}_p(X)\right] \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, $\widehat{ES}_p(X)$ is consistent.