Example 1

$X_1, \dots, X_n$ are IID $Exp(\lambda)$.

The MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = -\bar{x} \log(1-p)$$

Its bias is

$$Bias\left[\widehat{VaR}_p(X)\right] = E\left[-\bar{x} \log(1-p)\right] - \left(-\frac{1}{\lambda}\log(1-p)\right)$$

$$= \left[-E(\bar{x}) + \frac{1}{\lambda}\right] \log(1-p)$$

$$= \left[-E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{\lambda}\right] \log(1-p)$$

$$= \left[-\frac{1}{n}\sum_{i=1}^{n} E(x_i) + \frac{1}{\lambda}\right] \log(1-p)$$

$$= \left[-\frac{1}{n}\sum_{i=1}^{n} \frac{1}{\lambda} + \frac{1}{\lambda}\right] \log(1-p)$$

$$= \left[-\frac{1}{n}\cdot\frac{n}{\lambda} + \frac{1}{\lambda}\right] \log(1-p)$$

$$= 0$$

$\Rightarrow$ $\widehat{VaR}_p(X)$ is unbiased.

$$MSE\left[\widehat{VaR}_p(X)\right]$$

$$= MSE\left[-\bar{x}\log(1-p)\right]$$

$$= Var\left[-\bar{x}\log(1-p)\right]$$

$$= \left[\log(1-p)\right]^2 Var(\bar{x})$$

$$= \left[\log(1-p)\right]^2 Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \left[\log(1-p)\right]^2 \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i)$$

$$= \left[\log(1-p)\right]^2 \frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{\lambda^2}$$

$$= \left[\log(1-p)\right]^2 \frac{1}{n^2}\frac{n}{\lambda^2}$$

$$= \left[\log(1-p)\right]^2 \frac{1}{\lambda^2 n} \longrightarrow 0 \text{ as } n \to \infty$$

Hence, $\widehat{VaR}_p$ is consistent

Example 2 Suppose $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$. The MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \hat{\mu} + \hat{\sigma}\, \Phi^{-1}(p)$$

$$= \bar{x} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\, \Phi^{-1}(p)$$

$$\boxed{\begin{array}{l} \displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 \sim \sigma^2 \chi^2_{n-1} \\[2ex] \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \end{array}}$$

$$\text{Bias}\left[\widehat{VaR}_p(X)\right] = E\left[\widehat{VaR}_p(X)\right] - VaR_p(X)$$

$$= E\left[\bar{x} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\, \Phi^{-1}(p)\right]$$

$$- \left[\mu + \sigma\, \Phi^{-1}(p)\right]$$

$$= E(\bar{x}) + \frac{1}{\sqrt{n}} E\left(\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right) \Phi^{-1}(p)$$

$$- \left[\mu + \sigma\, \Phi^{-1}(p)\right]$$

$$= \mu + \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2}\, \Gamma(n/2)}{\Gamma((n-1)/2)} \Phi^{-1}(p)$$

$$- \mu - \sigma\, \Phi^{-1}(p)$$

$\rightarrow 0$ as $n \rightarrow \infty$.