Ex 1

Suppose X has the CDF

$F_X(x) = x^a$, $0 < x < 1$. Find VaR and ES.

$$F_X(x) = p$$

$$\Rightarrow \quad x^a = p$$

$$\Rightarrow \quad x = p^{\frac{1}{a}}$$

$$\Rightarrow \quad \boxed{VaR_p(X) = p^{\frac{1}{a}}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p t^{\frac{1}{a}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{1}{p}\left[\frac{p^{\frac{1}{a}+1}}{\frac{1}{a}+1} - 0\right]$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

Ex 2 Suppose X has the CDF

$$F_X(x) = \frac{1}{1+x^a}, \quad x > 0.$$

Set $F_X(x) = p$

$$\Rightarrow \frac{1}{1+x^a} = p$$

$$\Rightarrow 1 + x^a = \frac{1}{p}$$

$$\Rightarrow x^a = \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$\Rightarrow x = \left(\frac{1-p}{p}\right)^{\frac{1}{a}}$$

$$\Rightarrow \text{VaR}_p(X) = \left(\frac{1-p}{p}\right)^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left(\frac{1-t}{t}\right)^{\frac{1}{a}} dt$$

$$= \frac{1}{p}\int_0^p t^{-\frac{1}{a}} (1-t)^{\frac{1}{a}} dt$$

Incomplete beta function

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= \frac{1}{p} B_p\left(1-\frac{1}{a}, 1+\frac{1}{a}\right).$$

Example 3 Suppose X has the CDF $F_X(x) = \frac{1}{1+e^{-x}}$, Find VaR and ES.

$-\infty < x < \infty$

Set $F_X(x) = p$

$$\Rightarrow \frac{1}{1+e^{-x}} = p$$

$$\Rightarrow 1+e^{-x} = \frac{1}{p}$$

$$\Rightarrow e^{-x} = \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$\Rightarrow x = -\log\left(\frac{1-p}{p}\right)$$

$$\Rightarrow VaR_p(X) = -\log\left(\frac{1-p}{p}\right)$$

$$ES_p(X) = -\frac{1}{p}\int_0^p \log\left(\frac{1-t}{t}\right) dt$$

$$= -\frac{1}{p}\left[\int_0^p \log(1-t)\, dt - \int_0^p \log t \, dt\right]$$

$$= -\frac{1}{p}\left\{\left[\log(1-t) . t\right]_0^p + \int_0^p \frac{t}{1-t}\, dt - \left[\log t . t\right]_0^p + \int_0^p \frac{t}{t}\, dt\right\}$$

$$= -\frac{1}{p}\left\{\log(1-p) . p - 0 + \int_0^p \frac{(t-1)+1}{1-t}\, dt - \log p . p + 0 + p\right\}$$

$$= -\frac{1}{p}\left\{ \log(1-p)\cdot p + \Big[-t - \log(1-t)\Big]_0^p \right.$$

$$\left. - \log p \cdot p \right\}$$

$$= -\frac{1}{p}\left\{ \log(1-p)\cdot p - p - \log(1-p) \right.$$

$$\left. - \log p \cdot p \right\}$$

Ex 4 Suppose X has the CDF

$$F_X(x) = [x(2-x)]^a, \quad 0 < x < 1.$$

Find VaR and ES.

Set $[x(2-x)]^a = p$

$$\Rightarrow \quad x(2-x) = p^{\frac{1}{a}}$$

$$\Rightarrow \quad 2x - x^2 - p^{\frac{1}{a}} = 0$$

$$\Rightarrow \quad x^2 - 2x + p^{\frac{1}{a}} = 0$$

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{4 - 4p^{\frac{1}{a}}}}{2}$$

$$= 1 \pm \sqrt{1 - p^{\frac{1}{a}}}$$

$$\Rightarrow \quad x = 1 - \sqrt{1 - p^{\frac{1}{a}}}$$

$$\Rightarrow \quad \mathrm{VaR}_p(X) = 1 - \sqrt{1 - p^{\frac{1}{a}}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[1 - \sqrt{1 - t^{\frac{1}{a}}} \right] dt$$

$$= \frac{1}{p} \left[p - \int_0^p \sqrt{1 - t^{\frac{1}{a}}}\, dt \right]$$

Set $y = t^{\frac{1}{a}}$

$$t = y^a$$

$$\frac{dt}{dy} = a y^{a-1}$$

$$= 1 - \frac{1}{p} \int_0^{p^{\frac{1}{a}}} \sqrt{1-y}\; a y^{a-1} \, dy$$

$$= 1 - \frac{a}{p} \int_0^{p^{\frac{1}{a}}} y^{a-1} \sqrt{1-y}\, dy$$

$$= 1 - \frac{a}{p} B_{p^{\frac{1}{a}}}\left(a, \frac{3}{2}\right).$$