Example 1 Suppose a portfolio has 2 investments with losses X_1 & X_2. Let

$$\bar{F}_{X_1, X_2}(x_1, x_2) = (1 + x_1 + x_2)^{-a}$$

for $x_1 > 0$ and $x_2 > 0$. Find the distributions of S, U and V.

The joint PDF of (X_1, X_2) is

$$f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2}(1 + x_1 + x_2)^{-a}$$

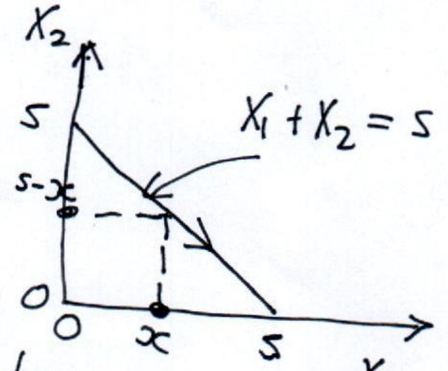
$$= \frac{\partial}{\partial x_1}(-a)(1 + x_1 + x_2)^{-a-1}$$

$$= (-a)(-a-1)(1 + x_1 + x_2)^{-a-2}$$

$$= a(a+1)(1 + x_1 + x_2)^{-(a+2)}$$

The PDF of S is

$$f_S(s) = \int_0^s f_{X_1, X_2}(x, s-x)\,dx$$



$$= \int_0^s a(a+1)(1 + \cancel{x} + s - \cancel{x})^{-(a+2)}\,dx$$

$$= a(a+1)(1+s)^{-(a+2)} \int_0^s 1\,dx$$

$$= \boxed{a(a+1)(1+s)^{-(a+2)}\, s}$$

The PDF of S' is

$$f_S(s) = a(a+1)\,(1+s)^{-(a+2)}\,s$$

The CDF of S' is

$$F_S(s) = \int_0^s f_S(t)\,dt$$

$$= a(a+1)\int_0^s (1+t)^{-(a+2)}\,t\,dt$$

$$\boxed{\begin{aligned} y &= \frac{t}{1+t} \\ \frac{1}{y} &= 1 + \frac{1}{t} \\ \frac{1}{y} - 1 &= \frac{1}{t} \\ t &= \frac{y}{1-y} \\ \frac{dt}{dy} &= \frac{1}{(1-y)^2} \end{aligned}}$$

$$= a(a+1)\int_0^{\frac{s}{1+s}} (1-y)^{a+2}\,\frac{y}{1-y}\,\frac{dy}{(1-y)^2}$$

$$= a(a+1)\int_0^{\frac{s}{1+s}} y\,(1-y)^{a-1}\,dy$$

$$\boxed{B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad \text{Incomplete beta function}}$$

$$= a(a+1)\,B_{\frac{s}{1+s}}(2, a)$$

The CDF of U is

$$F_U(u) = P(U \leq u)$$
$$= 1 - P(U > u)$$
$$= 1 - P(\min(X_1, X_2) > u)$$
$$= 1 - P(X_1 > u, X_2 > u)$$
$$= 1 - \bar{F}_{X_1, X_2}(u, u)$$
$$= 1 - (1 + u + u)^{-a}$$
$$= 1 - (1 + 2u)^{-a}$$

The PDF of U is

$$f_U(u) = \frac{d F_U(u)}{du} = a(1+2u)^{-a-1} \times 2$$
$$= 2a(1+2u)^{-a-1}$$

The CDF of V is

$$F_V(v) = P(V \leq v)$$
$$= P(\max(X_1, X_2) \leq v)$$
$$= P(X_1 \leq v, X_2 \leq v)$$
$$= F_{X_1, X_2}(v, v)$$
$$= 1 - \bar{F}_{X_1, X_2}(0, v) - \bar{F}_{X_1, X_2}(v, 0) + \bar{F}_{X_1, X_2}(v, v)$$
$$= 1 - (1+0+v)^{-a} - (1+v+0)^{-a} + (1+v+v)^{-a}$$
$$= 1 - 2(1+v)^{-a} + (1+2v)^{-a}.$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$
$$= 2a(1+v)^{-a-1} - 2a(1+2v)^{-a-1}$$

Example 2 Suppose a portfolio has 2 investments with losses X_1 & X_2.

Let
$$F_{X_1,X_2}(x_1,x_2) = \frac{1}{1+e^{-x_1}+e^{-x_2}}$$

Find the distributions of U & V.

The CDF of U is

$$\begin{aligned}
F_U(u) &= P(U \le u) \\
&= 1 - P(U > u) \\
&= 1 - P(\min(X_1,X_2) > u) \\
&= 1 - P(X_1 > u, X_2 > u) \\
&= 1 - \bar{F}_{X_1,X_2}(u,u) \\
&= 1 - \left[1 - F_{X_1,X_2}(u,\infty) - F_{X_1,X_2}(\infty,u) + F_{X_1,X_2}(u,u)\right] \\
&= 1 - \left[1 - \frac{1}{1+e^{-u}} - \frac{1}{1+e^{-u}} + \frac{1}{1+2e^{-u}}\right] \\
&= \frac{2}{1+e^{-u}} - \frac{1}{1+2e^{-u}}.
\end{aligned}$$

The PDF of U is

$$f_U(u) = +\frac{2e^{-u}}{(1+e^{-u})^2} - \frac{2e^{-u}}{(1+2e^{-u})^2}$$

The CDF of V is

$$F_V(v) = P(V \leq v)$$
$$= P(\max(X_1, X_2) \leq v)$$
$$= P(X_1 \leq v, X_2 \leq v)$$
$$= F_{X_1, X_2}(v, v)$$
$$= \frac{1}{1 + e^{-v} + e^{-v}}$$
$$= \frac{1}{1 + 2e^{-v}}.$$

The PDF of V is

$$f_V(v) = \frac{2e^{-v}}{(1 + 2e^{-v})^2}.$$