Ex 1

Suppose $X_1, \ldots, X_n$ are IID Exp(a).
Find the CDF & PDF of U & V.

The CDF of U is

$$
\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= 1 - P(U > u) \\
&= 1 - P(\min(X_1, \ldots, X_n) > u) \\
&= 1 - P(X_1 > u, \ldots, X_n > u) \\
&\overset{\text{indep}}{=} 1 - P(X_1 > u) \cdots P(X_n > u) \\
&= 1 - [1 - P(X_1 \leq u)] \cdots [1 - P(X_n \leq u)] \\
&= 1 - [1 - (1 - e^{-au})] \cdots [1 - (1 - e^{-au})] \\
&= 1 - e^{-au} \cdots e^{-au} \\
&= 1 - e^{-nau}
\end{aligned}
$$

The PDF of U is

$$f_U(u) = nae^{-nau}.$$

The CDF of V is

$$F_V(v) = P(V \leq v)$$

$$= P(\max(X_1, \ldots, X_n) \leq v)$$

$$= P(X_1 \leq v, \ldots, X_n \leq v)$$

$$\stackrel{\text{indep}}{=} P(X_1 \leq v) \cdots P(X_n \leq v)$$

$$= (1 - e^{-av}) \cdots (1 - e^{-av})$$

$$= (1 - e^{-av})^n$$

The PDF of V is

$$f_V(v) = n(1 - e^{-av})^{n-1} a e^{-av}$$

Ex 2 Suppose $X_1, \ldots, X_N$ are IID $Exp(a)$ where $N \sim Geom(p)$. Assume $X_1, \ldots$ and N are independent. Find the CDF & PDF of U & V.

The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= 1 - P(U > u)$$

$$= 1 - P(\min(X_1, \ldots, X_N) > u)$$

$$= 1 - \sum_{n=1}^{\infty} P(\min(X_1, \ldots, X_N) > u \mid N=n) P(N=n)$$

total probability rule

$$= 1 - \sum_{n=1}^{\infty} P(\min(X_1, \ldots, X_n) > u) \, p(1-p)^{n-1}$$

$$= 1 - \sum_{n=1}^{\infty} P(X_1 > u, \ldots, X_n > u) \, p(1-p)^{n-1}$$

$$\overset{indep}{=} 1 - \sum_{n=1}^{\infty} P(X_1 > u) \cdots P(X_n > u) \, p(1-p)^{n-1}$$

$$= 1 - \sum_{n=1}^{\infty} (1-(1-e^{-au})) \cdots (1-(1-e^{-au})) \, p(1-p)^{n-1}$$

$$= 1 - \sum_{n=1}^{\infty} e^{-nau} \, p(1-p)^{n-1}$$

$$= 1 - \sum_{n=1}^{\infty} e^{-(n-1+1)au} \, p(1-p)^{n-1}$$

$$= 1 - e^{-au} p \sum_{n=1}^{\infty} \left[e^{-au} (1-p) \right]^{n-1}$$

$\boxed{\text{Set } m = n-1}$

$$= 1 - e^{-au} p \sum_{m=0}^{\infty} \left[e^{-au} (1-p) \right]^{m}$$

$$\boxed{\sum_{m=0}^{\infty} x^{m} = \frac{1}{1-x}}$$

$$= 1 - \frac{e^{-au} p}{1 - e^{-au} (1-p)}$$

$$= \frac{1 - e^{-au}}{1 - e^{-au} (1-p)}.$$

The PDF of U is

$$f_U(u) = \left\{ \left[1 - e^{-au}(1-p) \right] a e^{-au} - \left[1 - e^{-au} \right] a (1-p) e^{-au} \right\} / \left[1 - e^{-au} (1-p) \right]^2.$$

The CDF of V is

$$F_V(v) = P(V \leq v)$$

$$= P(\max(X_1, \dots, X_N) \leq v)$$

$$= \sum_{n=1}^{\infty} P(\max(X_1, \dots, X_N) \leq v \mid N = n) P(N = n)$$

total prob rule

$$= \sum_{n=1}^{\infty} P(\max(X_1, \dots, X_n) \leq v) \, p(1-p)^{n-1}$$

$$= \sum_{n=1}^{\infty} P(X_1 \leq v, \dots, X_n \leq v) \, p(1-p)^{n-1}$$

$$\overset{\text{indep}}{=} \sum_{n=1}^{\infty} P(X_1 \leq v) \cdots P(X_n \leq v) \, p(1-p)^{n-1}$$

$$= \sum_{n=1}^{\infty} (1 - e^{-av}) \cdots (1 - e^{-av}) \, p(1-p)^{n-1}$$

$$= \sum_{n=1}^{\infty} (1 - e^{-av})^n \, p(1-p)^{n-1}$$

$$= (1 - e^{-av}) p \sum_{n=1}^{\infty} \left[(1 - e^{-av})(1-p)\right]^{n-1}$$

$\boxed{\text{set} \quad m = n-1}$

$$= (1 - e^{-av}) p \sum_{m=0}^{\infty} \left[(1 - e^{-av})(1-p)\right]^m$$

$$= (1 - e^{-av}) p \times \frac{1}{1 - (1 - e^{-av})(1-p)}$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = \Big\{ \left[1 - (1-e^{-av})(1-p)\right] p a e^{-av} - (1-e^{-av})\, p(1-p)\, a e^{-av} \Big\} \Big/ \left[1-(1-e^{-av})(1-p)\right]^2$$

Example 3 Suppose $X_1, \ldots, X_n$ are $Exp(a_i)$, $i = 1, \ldots, n$ and are independent. Find the CDF & PDF of U & V.

The CDF of U is

$$F_U(u) = P(U \le u)$$
$$= 1 - P(U > u)$$
$$= 1 - P(\min(X_1, \ldots, X_n) > u)$$
$$= 1 - P(X_1 > u, \ldots, X_n > u)$$
$$\stackrel{\text{indep}}{=} 1 - P(X_1 > u) \cdots P(X_n > u)$$
$$= 1 - (1 - P(X_1 \le u)) \cdots (1 - P(X_n \le u))$$
$$= 1 - (1 - (1 - e^{-a_1 u})) \cdots (1 - (1 - e^{-a_n u}))$$
$$= 1 - e^{-a_1 u} \cdots e^{-a_n u}$$
$$= 1 - e^{-(a_1 + \cdots + a_n) u}$$

The PDF of U is

$$f_U(u) = (a_1 + \cdots + a_n) e^{-(a_1 + \cdots + a_n) u}.$$